On the Type E case of the Brown–Wemyss realization conjecture

Shutao Bi*

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Abstract

We consider the remaining Type E case of the Brown–Wemyss numerical realization conjecture. The noncommutative splitting lemma and the Type A and D classifications of Brown and Wemyss leave two-generator cyclic potentials with cubic part x^3. For these remaining potentials, we construct geometric realizations under the hypothesis of radical growth dimension at most one. The construction combines marked ADE deformations and cyclic coordinate changes for lengths three, four, and six with a filtered finiteness argument in length five. Under the numerical growth hypothesis, cubic lower bounds exclude the other branches for every choice of formal tail. The resulting contractions have a regular source and a complete local cDV target. Finite-dimensional algebras are realized by threefold flops, and the sole growth-one algebra in this residual case is $\mathrm{Jac}(x^3 + xy^3)$, for which we give a divisor-to-curve contraction.

*Email: shutaobi@mail.sdu.edu.cn. Homepage: https://shutaobi.github.io/.

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1 Introduction

For a flopping curve C on a threefold, the contraction algebra represents the noncommutative deformation functor of $\mathcal{O}_C(-1)$ [Donovan–Wemyss 16, Theorem 1.2]. The realization problem asks which completed associative algebras arise geometrically as contraction algebras of birational crepant contractions. Brown and Wemyss formulated a numerical criterion for realizing completed Jacobi algebras, expressed through the growth of their radical quotients [Brown–Wemyss 25, Conjecture 1.11]. This criterion encompasses both flops, whose contraction algebras are finite-dimensional, and contractions of a divisor to a curve, whose contraction algebras have radical growth dimension one.

The splitting lemma and the Type A and D classifications established by Brown and Wemyss leave two-generator cyclic potentials whose cubic part is x^3 [Brown–Wemyss 25, Corollaries 1.12 and 8.18]. Our contribution concerns this remaining Type E case. We use the low-degree terms of the potential to determine the geometric construction, and retain an arbitrary formal tail throughout the argument.

For a finite set of generators, we set

$$F_d = \mathbb{C}\langle\langle x_1, \dots, x_d \rangle\rangle, \qquad \mathfrak{m}_d = (x_1, \dots, x_d), \qquad (F_d)_{\mathrm{cyc}} = F_d/\overline{[F_d, F_d]}.$$

The overline in the cyclic quotient denotes the topological closure of the linear commutator subspace. Throughout this paper, every derivative ideal is understood to be a closed two-sided ideal. For a cyclic potential φ, we write

$$A_\varphi = \mathrm{Jac}(\varphi) = F_d/\overline{(\partial_{x_1}\varphi, \dots, \partial_{x_d}\varphi)}, \qquad J_\varphi = \mathrm{rad}(A_\varphi).$$

For an augmented complete local algebra A with Jacobson radical J, the radical growth dimension is defined by

$$\mathrm{Jdim}\, A = \inf\left\{r \geq 0 : \dim_{\mathbb{C}}(A/J^n) \leq C_r n^r \text{ for every } n \geq 1 \text{ and some } C_r > 0\right\}.$$

When the potential is fixed, we write A for its Jacobi algebra and J for its Jacobson radical. We write O_q for terms of ordinary word length at least q, indicating weighted orders explicitly when they are used.

Definition 1.1. Write $\mathrm{Geo}(A)$ if there is a projective birational crepant morphism $\pi : X \to \mathrm{Spec}\, R$, where R is a three-dimensional complete local normal cDV complex algebra, X is regular, and the reduced closed fiber is $\mathbb{P}^1$, together with a continuous complete-algebra isomorphism $A \cong A_{\mathrm{con}}(\pi)$. Regularity is the smooth-source convention in this formal local setting. The generator used to define A_{con} is normalized and basic.

Conjecture 1.2 (Brown–Wemyss Numerical Realization)**.** *For every $d \geq 1$ and every cyclic formal potential $\varphi \in \mathfrak{m}_d^2$, the condition* $\mathrm{Jdim}\,\mathrm{Jac}(\varphi) \leq 1$ *implies* $\mathrm{Geo}(\mathrm{Jac}(\varphi))$.

A constant term in φ has zero cyclic derivatives. If the linear part of φ is nonzero, one of its cyclic derivatives contains a nonzero constant term and is therefore a unit, which makes the completed Jacobi algebra vanish. We may thus restrict attention to potentials of order at least two.

The Type E arguments below, together with the published splitting and Type A and D realization results, give the following statement.

Theorem 1.3 (Geometric Realization)**.** *Let φ be a cyclic formal potential in finitely many variables, with no linear term, and let its nonzero augmented Jacobi algebra satisfy* $\mathrm{Jdim}\,\mathrm{Jac}(\varphi) \leq 1$. *Then* $\mathrm{Jac}(\varphi)$ *is isomorphic to the normalized basic contraction algebra of a projective birational crepant contraction with regular source, complete local normal three-dimensional cDV target, and reduced closed fiber $\mathbb{P}^1$. The finite-dimensional case is flopping; the growth-one case is divisor-to-curve. In residual Type E the only growth-one algebra is* $\mathrm{Jac}(x^3 + xy^3)$.

Here residual Type E refers to the potentials $x^3 + O_4$ left by the published reduction. The proof divides these potentials according to their quartic terms and, when necessary, their fifth and sixth terms.

For lengths three, four, and six, we put the potential into a sum of one-variable series in three linear forms. The Jacobi relations identify the three one-variable derivatives with a common central element. Preparing the three resulting equations produces a formal arc in the marked ADE deformation space. The first derivatives of the constant coefficients of these polynomials determine regularity of the source, and the full endomorphism algebra identifies the contraction quotient. These constructions occupy Section 2.

For length five, Section 3 first establishes finite-dimensionality. If the Jacobi algebra were infinite-dimensional, its associated graded algebra would have generic rank 25 over a central polynomial ring. The determinant of right multiplication on a finite free module would then force this rank to be divisible by 4. Finiteness then permits polynomial truncation and the application of Wemyss's gluing construction.

Section 4 supplies the exclusions needed to make this case division exhaustive under the growth hypothesis. A vanishing reduced quartic, or either degenerate sextic axis, gives a cubic lower bound for every higher tail. Section 5 combines the constructions with these exclusions. The finite exact calculations used in the proof are described in Section 6.

Relation to earlier work

The subsequent classification of contraction algebras by Brown and Wemyss also begins with low-order normal forms [Brown–Wemyss 26, Section 3]. Their criterion uses *algebraic length*. They require a central element such that quotienting the Jacobi algebra by the closed ideal it generates gives an algebra isomorphic to the corner $e\Pi e$, where Π is an ADE preprojective algebra and e is a primitive idempotent. The marked vertex corresponding to e determines the algebraic length [Brown–Wemyss 26, Definition 1.1, Corollary 8.14 and Theorem 9.2]. The numerical hypothesis concerns only the dimensions of the radical quotients.

For the family $x^3 + xy^3 + p(y)$, Pan constructed a smooth flop for every nonzero tail $p \in y^4\mathbb{C}[[y]]$ and a divisorial contraction at $p = 0$ [Pan 26, Proposition 4.11 and Corollary 4.12]. Brown and Wemyss gave another divisor-to-curve realization of $\mathrm{Jac}(x^3 + xy^3)$ [Brown–Wemyss 26, Theorem 1.7]. The constructions here use the marked deformation spaces of Katz and Morrison [Katz–Morrison 92, Sections 5–6], Karmazyn's presentations of the universal flopping algebras [Karmazyn 19, Theorem 1.3 and Examples 4.24–4.27], and Van den Bergh's tilting generators [Van den Bergh 04, Propositions 3.2.6–3.2.7]. The quotient of the endomorphism algebra by maps factoring through the structure sheaf summand yields the contraction algebra.

At length five, the geometric input is the fifth family in Wemyss's gluing theorem [Wemyss 26, Section 1.6 and Theorem 1.7]. Brown and Wemyss identified the deformation algebras of the corresponding curve neighborhoods using free necklace potentials [Brown–Wemyss 24, Corollary 6.4]. The proof of Proposition 3.8 combines the filtered finiteness argument, polynomial truncation and this gluing construction. The same subsection gives a second route through the later normal form of [Brown–Wemyss 26, Theorem 8.3].

2 Cyclic normal forms and geometric constructions

The published reduction of Brown and Wemyss leaves a cyclic cubic x^3 and an arbitrary higher tail. After separating the quartic orbits, we construct the length-three, length-four, and length-six families. The alternating quartic and the growth exclusions are treated in the next two sections.

2.1 Reduction to two generators

Proposition 2.1 (Published reduction)**.** *Conjecture 1.2 is equivalent to its two-generator restriction*

$$\varphi = x^3 + O_4, \qquad \operatorname{Jdim}\mathrm{Jac}(\varphi) \leq 1.$$

Cases of corank at most one and two-generator cases with $\dim J^2/J^3 = 2$ *are already geometric in both growth dimensions.*

Proof. The noncommutative splitting lemma removes the quadratic variables without altering the isomorphism class of the completed Jacobi algebra [Brown–Wemyss 25, Lemma 4.5 and Appendix A]. Under the growth hypothesis $\operatorname{Jdim}\mathrm{Jac}(\varphi) \leq 1$, the corank bound established by Brown and Wemyss leaves $\dim J/J^2$ at most two. In corank zero the algebra is $\mathbb{C}$, and in corank one it is $\mathbb{C}[[y]]$ or $\mathbb{C}[[y]]/(y^k)$.

In corank two the cubic part is nonzero. Up to a linear change of variables, this cubic form is $x^3 + y^3$, x^2y, or x^3. The first two cubics give a second radical layer of dimension two, whereas the third cubic gives a second radical layer of dimension three. The Type A and D constructions in [Brown–Wemyss 25, Sections 5–6 and Corollary 8.18] realize all the cases covered there in the sense of Definition 1.1, including their infinite-dimensional forms. Geometric realization is invariant under continuous algebra isomorphism. It therefore remains to treat the third cubic with an arbitrary higher tail, and the converse follows by specialization. □

2.2 The five quartic orbits

A necklace is a cyclic equivalence class of words in the free completed algebra. In each cyclic reduction step, a necklace is called good if all of its cyclic shifts avoid the forbidden subwords, including across the cyclic boundary, and bad otherwise.

Lemma 2.2 (Removal of cyclic xx words)**.** *Every $x^3 + O_4$ is cyclically right-equivalent, by a continuous unitriangular automorphism fixing y, to a series whose higher necklaces contain no adjacent xx, including at the cyclic boundary. Its quartic part is*

$$a(xy)^2 + bxy^3 + cy^4. \tag{2.1}$$

Proof. At degree $n \geq 4$, we rotate each necklace containing adjacent letters xx to the form x^2h, where h has degree $n-2$. We write the sum of these terms in degree n as $[x^2h_n]$. The substitution $x \mapsto x - h_n/3$, with y held fixed, changes the cyclic cubic $[x^3]$ by $-[x^2h_n]$ in this degree. Multiple replacements in the cubic and replacements in terms of degree at least four have degree greater than n. Each elimination step therefore preserves all previous cancellations.

The successive substitutions and their inverses stabilize on every finite word quotient $F_2/\mathfrak{m}_2^k$. Their inverse limit yields the required continuous unitriangular automorphism. The three quartic necklaces in (2.1) exhaust those with no adjacent letters xx in cyclic order. By the cyclic chain rule, each cyclic derivative of the transformed potential lies in the image of the closed derivative ideal. Applying the inverse substitution gives equality of the derivative ideals under transport. □

Proposition 2.3 (Five quartic orbits)**.** *Put $\Delta = b^2 - 8ac$. A linear map preserving the cubic has the form $x \mapsto \zeta x$, $y \mapsto tx + vy$, where $\zeta^3 = 1$ and $v \neq 0$. Its reduced quartic coefficients are*

$$a' = v^2(\zeta^2 a + \zeta t b + 2t^2 c), \qquad b' = v^3(\zeta b + 4tc), \qquad c' = v^4 c, \tag{2.2}$$
$$\Delta' = \zeta^2 v^6 \Delta.$$

Allowing also a nonzero scalar multiple of the potential, the five four-jet representatives are

$$x^3, \quad x^3 + (xy)^2, \quad x^3 + xy^3, \quad x^3 + y^4, \quad x^3 + (xy)^2 + y^4.$$

Their strata are respectively the zero quartic, $c = b = 0, a \neq 0$, $c = 0, b \neq 0$, $c \neq 0, \Delta = 0$, and $c \neq 0, \Delta \neq 0$.

Proof. We expand the transformed quartic and remove all cyclic xx terms using Lemma 2.2. For the contribution of the quartic coefficient b, expand $\zeta x(tx + vy)^3$ and consider the terms linear in t. Only the middle placement of the new x factor avoids adjacent letters xx and contributes to $(xy)^2$, with coefficient $v^2\zeta tb$. The expansion of y^4, after multiplication by its coefficient, contributes the two words $xyxy$ and $yxyx$ to the same cyclic class, giving the total coefficient $2v^2t^2c$. Together with the direct contributions to the other necklaces, this establishes (2.2) and the stated transformation law $\Delta' = \zeta^2 v^6 \Delta$.

The orbit classification follows from these coefficient formulas. For $c = 0, b \neq 0$, we choose $t = -\zeta a/b$ to eliminate a'. For $c \neq 0$, we choose $t = -\zeta b/(4c)$ to eliminate b'. The remaining alternating coefficient is then proportional to $a - b^2/(8c) = -\Delta/(8c)$. Rescaling the variables and multiplying the potential by a nonzero scalar yield the five four-jet representatives. Multiplying the potential by a scalar preserves the closed derivative ideal. □

Remark 2.4. Each representative specifies only a four-jet. The higher terms in $\mathfrak{m}_2^5$ remain arbitrary.

2.3 A realization criterion

Let $P, Q, R \in u\mathbb{C}[[u]]$, and let U, V, W be their antiderivatives with zero constant term. The following criterion is applied in each of the length-three, length-four, and length-six constructions.

Theorem 2.5 (Realization of the three-direction families)**.** *Suppose the exact orders of P, Q, R are one of*

$$(3, 3, 2), \qquad (2, 4, 3), \qquad (2, 3, 5),$$

with nonzero leading coefficients. The Jacobi algebra of $U(x) + V(y) + W(-x - y)$ is geometric. The marked lengths are respectively three, four and six. The latter two are small and finite-dimensional. In length three the divisorial alternative is given by (2.3) on the prepared arc.

The parameter of the arc will be the common central value of $P(x), Q(y), R(-x - y)$. We first establish the geometric properties of an arc and the endomorphism comparison. Subsection 2.4 then constructs the arc by one-variable preparation and completes the proof of Theorem 2.5.

For each length $\ell \in \{3, 4, 5, 6\}$, write $\mathbb{H}_\ell$ for Karmazyn's marked parameter ring, $\mathcal{Y}_\ell \to \operatorname{Spec} R_\ell$ for the simultaneous partial resolution, and Λ_ℓ for the full algebra [Karmazyn 19, Theorem 1.3 and Section 3.3]. Let e_0 denote the projector onto the structure sheaf summand in Λ_ℓ. The universal contraction quotient is

$$C_\ell = \Lambda_\ell / \Lambda_\ell e_0 \Lambda_\ell.$$

Take an arc $\nu : \mathbb{H}_\ell \to \mathbb{C}[[w]]$ through the marked origin. We pull the family back along the arc, localize and complete its affine target at the central singular point, and base change the projective source to that target.

Lemma 2.6 (Endomorphisms and contraction quotients)**.** *For these marked families, the full endomorphism algebra commutes with arc base change and target completion. If the resulting source is regular and the morphism is crepant with reduced closed fiber $\mathbb{P}^1$, its normalized basic contraction algebra is isomorphic to the completed e_0-quotient.*

Proof. The comparison requires ordinary endomorphism base change, identification of the basic generator, and passage to the projector quotient.
(i) Endomorphisms under base change. We write $\Lambda_\ell = \mathrm{End}(E)^{\mathrm{op}} = \mathrm{End}(E^\vee)$ for Karmazyn's specified tilting bundle $E = \mathcal{O} \oplus N$ [Karmazyn 19, Theorem 3.3 and Section 3.3]. Both the universal source and the full algebra are flat over $\mathbb{H}_\ell$. On a finite affine open cover of the separated source, the Čech complex for $\mathcal{E}nd(E)$ is bounded with flat terms. Its only nonvanishing cohomology is the flat endomorphism algebra in degree zero. Descending from the last term shows that every boundary module is flat. Tensoring the complex with the arc ring therefore computes the ordinary endomorphisms after pullback and gives zero higher self-Ext.

Flat base change on the target gives the same comparison after localization and completion. Generation follows from adjunction and the conservativity of pushforward along the affine base-change map to the original source.

(ii) The generator on the contraction. On a crepant arc with regular source, rationality of the cDV target gives $R\pi_*\mathcal{O}_X = \mathcal{O}_{\mathrm{Spec}\, R}$, and the fibers have dimension at most one. The dual bundle $P = E^\vee$ retains its global generation, its relatively ample determinant, and its structure sheaf summand. We have $H^1(P^\vee) = H^1(E) = 0$ because $E = \mathcal{H}om(\mathcal{O}, E)$ is a summand of $\mathcal{E}nd(E)$. The criterion of [Van den Bergh 04, Propositions 3.2.6–3.2.7] now identifies P as a projective generator for perversity minus one and E as a generator for perversity zero.

It remains to verify the multiplicities of the summands. Modulo w they reduce to the two distinct indecomposable summands of the basic generator on the completed central partial resolution. Since w belongs to the Jacobson radical of the full algebra over the complete local target, a nonzero idempotent cannot reduce to zero. A nontrivial direct-sum decomposition upstairs would therefore produce a nontrivial decomposition on the central surface. An isomorphism between the projective summands would likewise descend to that surface. Thus both summands occur with multiplicity one, and the structure sheaf summand still specifies the free projector.

(iii) The quotient and its completion. For a commutative scalar extension $H \to S$ and a full H-algebra Λ, the image of the multiplication map $\Lambda e_0 \otimes_H e_0\Lambda \to \Lambda$ is the two-sided ideal $\Lambda e_0 \Lambda$. Forming its cokernel commutes with arbitrary scalar extension, yielding

$$(\Lambda/\Lambda e_0\Lambda) \otimes_H S \cong (\Lambda \otimes_H S)\big/(\Lambda \otimes_H S)e_0(\Lambda \otimes_H S).$$

To compare the completion topologies, let $\mathfrak{n}$ be the maximal ideal of the completed target and J the Jacobson radical of the full algebra. We have $\mathfrak{n}\Lambda \subseteq J$. In the finite-dimensional algebra modulo $\mathfrak{n}$, the radical is nilpotent, giving $J^h \subseteq \mathfrak{n}\Lambda$ for some h. The two filtrations are therefore cofinal. The projector ideal is a finite submodule and is closed for this topology. Hence the geometric quotient agrees with the augmentation-complete quotient in the loop presentation. See also [Bi 26, Section 2.2] for the endomorphism comparison. □

Lemma 2.7 (The source first jet)**.** *For each residual A_{m-1} component, write its orthogonal projection polynomial as $p_m(Z, w) = Z^m + \sum_{i=0}^{m-2} a_i(w)Z^i$. The arc source is regular along the central fiber if and only if $a_0'(0) \neq 0$ at every such component.*

Proof. At each residual surface singularity along the central fiber, the orthogonal projection of Katz and Morrison yields the local completed equation $UV = p_m(Z, w)$ [Katz–Morrison 92, Section 6, Theorem 3]. The linear part of this defining equation is $-a_0'(0)w$. When $a_0'(0) \neq 0$, the formal implicit function theorem allows us to eliminate w, leaving the three regular parameters U, V, Z. If $a_0'(0) = 0$, the linear part vanishes identically, and the three-dimensional completed local ring has embedding dimension four, making the point nonregular.

Away from these residual singularities, the central surface is smooth, and flatness over the regular one-dimensional arc ring ensures regularity at all remaining points. The image of the nonregular locus in the completed local target is closed by projectivity. If nonempty, that image contains the closed point. Regularity along the closed fiber therefore gives regularity on the entire source. □

Proposition 2.8 (Geometry of the smooth arcs)**.** *The stated first-jet condition gives a projective birational crepant contraction with regular source, complete local normal cDV hypersurface target and reduced closed fiber $\mathbb{P}^1$. In the primitive length-four, length-five, and length-six families, the contraction is small. In length three, use the quotient parameters*

$$\beta^3 = B_1\beta + B_0, \quad \gamma^3 = C_1\gamma + C_0, \quad \delta^2 = D_0, \quad \beta + \gamma + \delta = t_K/3.$$

With B_0', C_0', D_0' nonzero at zero, it is divisorial precisely when

$$t_K = 0, \qquad B_1 = C_1, \qquad B_0 = -C_0. \tag{2.3}$$

Proof. Regularity of the source is supplied by Lemma 2.7. The distinction between the small and divisorial contraction types is governed by whether any positive roots of the ambient ADE root system vanish identically along the formal arc.

(i) The target and the closed fiber. The target space $\operatorname{Spec} R$ is flat over $\mathbb{C}[[w]]$, and its central Cartier section is a Du Val surface singularity. Hence the target is a three-dimensional cDV hypersurface. To verify that R is an integral domain, divide two nonzero elements by their respective maximal powers of w. Because the central Du Val surface is an integral domain, the reductions of these factored elements have a nonzero product in the central fiber, which guarantees that their product in R is nonzero. At every smooth point of the central surface, the total space is regular. A codimension-one singular component would have dimension two. Its intersection with the central Cartier section would be positive-dimensional, contradicting the isolated singularity of the Du Val surface. This establishes regularity in codimension one. As a hypersurface, the target automatically satisfies Serre's condition S_2, and is therefore normal.

The simultaneous partial resolution is projective and restricts to an isomorphism over the smooth locus of the surface. Its scheme-theoretic closed fiber is preserved under arc base change and completion of the target. On the central marked surface, the reduction of this fiber is $\mathbb{P}^1$, and its scheme-theoretic multiplicity is the coefficient ℓ of the retained simple root in the highest root of the ambient Dynkin diagram [Katz–Morrison 92, Sections 5–6]. By properness, every connected component of the regular source meets this irreducible closed fiber. Consequently, the source is integral, and the isomorphism over the dense smooth locus establishes birationality of the contraction.

(ii) Roots on the generic surface. Over the Laurent field $K = \mathbb{C}((w))$, the exceptional curves of the partial resolution are determined by the vanishing roots of the surface deformation. A vanishing root with nonzero coefficient at the retained vertex corresponds to an exceptional curve on the split generic surface. The root calculation below determines when this occurs. In part (iv) we check the closures at the completed target and deduce the contraction type.

For each deleted Dynkin chain of type A_{m-1}, let $z_1, \dots, z_m$ denote the centered roots of its projection polynomial. The first-jet condition $a_0'(0) \neq 0$ ensures that this projection polynomial is Eisenstein in w. Setting $w = q^m$, substituting $Z = qz$, and dividing by q^m, the polynomial reduces modulo q to $z^m = c$ with $c \neq 0$. By Hensel's lemma, each simple root lifts uniquely to the Puiseux series ring. The leading terms of the roots z_j in cyclic order are given by $c^{1/m}\zeta_m^j w^{1/m}$, where ζ_m is a primitive m-th root of unity. The inertia group of the field extension permutes these m roots cyclically. In particular, every Levi root is an entry difference $z_i - z_j$ with $i \neq j$, which has a nonzero leading term and therefore cannot vanish identically along the arc.

Let $k > 0$ be the coefficient of the retained simple root in an ambient positive root. The evaluation of this root along the arc decomposes as the sum of a base contribution in K and one contribution $\sum_j a_j z_j$ from each deleted Dynkin chain. The coefficients a_j are integers determined up to a common additive integer. If the chain attaches to the retained vertex at node h, these coefficients satisfy the congruence

$$\sum_j a_j \equiv -kh \pmod{m}.$$

On this chain, the retained simple root pairs to -1 at the attachment node h and to zero at all other nodes, so its restriction has weight $-\omega_h$. On a centered list of roots, the fundamental weight ω_h is represented by the sum of h coordinate entries. Adding a Levi root alters the coefficient vector by a difference of coordinate basis vectors. This yields the congruence, with identical divisibility tests regardless of the orientation of the chain.

Suppose an arm degree m is prime and coprime to every other arm degree. If an ambient root vanishes identically along the arc, the contribution of this arm must lie in K, which is the intersection of its Puiseux field with the compositum of the other arm fields. Because the root list is centered, the Galois average of this contribution over K is zero. Hence the contribution vanishes identically. The coefficient of $w^{1/m}$ in this contribution must therefore satisfy $\sum_j a_j \zeta_m^j = 0$. For prime m, the minimal polynomial of a primitive m-th root of unity forces all coefficients a_j to be equal, which implies $m \mid kh$. For $m = 4$, the vanishing of the leading coefficient forces $a_1 = a_3$ and $a_2 = a_4$, which requires their sum to be even. Applying these divisibility and parity criteria to the marked Dynkin diagrams of [Karmazyn 19, Examples 4.24–4.27] yields the conditions in the table below, where $1 \leq k \leq \ell$.

ℓ	arm degrees	necessary conditions for persistence
3	$(2,3,3)$	$2 \mid k$, so $k = 2$
4	$(2,3,4)$	$3 \mid k$, so $k = 3$; the quartic leading term survives
5	$(4,5)$	$5 \mid k$, so $k = 5$; the quartic leading term survives
6	$(2,3,5)$	$2 \mid k$, $\quad 3 \mid k$, $\quad 5 \mid k$

The quartic chain in lengths four and five attaches at an end node. An odd coefficient k yields an odd sum of coefficients, so the leading term of order $w^{1/4}$ survives. In length four, no other arm produces terms of that fractional order. In length five, orienting the quintic chain with $h = 2$, which is coprime to five, forces its contribution to vanish, leaving the quartic term uncancelled. In length six, no permissible coefficient $1 \leq k \leq 6$ meets all three divisibility tests simultaneously. This excludes persistent roots in lengths four through six.

(iii) The nine roots in length three. In the vertex numbering of [Karmazyn 19, Example 4.24], the retained node 6 is adjacent to the nodes $1, 3, 5$. To identify the candidate persistent roots, we begin with the root $2\alpha_6 + \alpha_1 + \alpha_3 + \alpha_5$ belonging to the central D_4 subdiagram. When ambient roots are normalized to have squared norm two, the projections of this root to the two A_2 arms are extremal fundamental weights of squared norm $2/3$, while its projection to the A_1 arm vanishes.

Every other root with coefficient $k = 2$ differs from this representative by an element of the Levi lattice, so its projection orthogonal to the Levi span is fixed. Within each relevant A_2 weight coset, the three permutations of the fundamental weight are precisely the centered triples of minimal norm. The chosen representative attains this minimum simultaneously in each arm. Any other ambient root of the same total norm must likewise attain the minimum in each arm. Permuting these triples independently yields all nine roots in a single Levi orbit.

Let b_i and c_j denote the roots of the two depressed cubics in the statement of the proposition. In these centered coordinates, the vanishing conditions for the nine roots in the orbit take the form

$$-b_i - c_j = 2t_K/3.$$

Let t_i denote the simple-root values in Karmazyn's coordinate system. Substituting $t_K = t_0 + t_1/2$ and $3t_6 = -(t_0+2t_1+t_2+2t_3+t_4+2t_5)$ into the representative yields the expression $-2t_K/3-(2t_2+t_3)/3-(2t_4+t_5)/3$. Here the last two terms are coordinate entries of the centered cubic lists. Applying Levi permutations generates the remaining eight equations.

Averaging any one of these nine equations over the action of cubic inertia forces $t_K = 0$. Because the cubics are irreducible, the existence of a single opposite pair forces the two root sets to be negatives of each other. Comparing coefficients of the polynomials gives the divisorial conditions (2.3). Conversely, when (2.3) holds, we obtain three opposite pairs. The three corresponding roots are mutually orthogonal. For distinct pairs both entries change, and within each fundamental triple the inner product drops by one from the squared norm. The overall pairing between distinct roots drops from two to zero, producing the generic singularity configuration A_1^3.

(iv) The contraction type and crepancy. When no ambient root persists, the generic surface is smooth. All exceptional curves of the contraction are then confined to the central fiber, which means that the morphism is small. The target is normal and Gorenstein, and the absence of exceptional prime divisors implies that the contraction is crepant.

In the exceptional length-three case where (2.3) holds, the closures of the three generic root curves on a splitting trait map onto curve germs in the target. To verify that these curve germs survive target completion, consider the relative critical algebra of the ADE surface equation. Its central fiber is the finite Jacobian algebra of the Du Val singularity. Homogeneous lifts of a basis generate the critical algebra over the polynomial marked base by induction on weight. The unique support point of the central fiber is the vertex singularity. A split critical point therefore has integral coordinates on its trait and specializes to the vertex, ensuring that the morphism extends to the completion.

Each exceptional divisor dominates the parameter curve because the central source surface is integral and maps birationally to its image. On the generic surface, adjunction guarantees that the canonical divisor has zero intersection with each exceptional (-2)-curve. Because the intersection matrix of the configuration is negative definite, all discrepancies vanish over the splitting field and consequently after Galois descent. This establishes crepancy. For the surface-deformation and root descriptions used in this argument, we refer to [Katz–Morrison 92, Sections 5–6]. □

Remark 2.9. Flatness of the full algebra need not pass to its projector quotient. For example, the $\mathbb{C}[[w]]$-free order $\left(\begin{smallmatrix} \mathbb{C}[[w]] & \mathbb{C}[[w]] \\ w\mathbb{C}[[w]] & \mathbb{C}[[w]] \end{smallmatrix}\right)$ has free-vertex quotient $\mathbb{C}[[w]]/(w)$. Likewise, a nonzero finite local algebra on which a radical central element acts nilpotently cannot be flat over its parameter DVR. This is why we verify that endomorphism formation commutes with base change on the full algebra before passing to the contraction quotient.

2.4 Construction of the arc from the potential

Lemma 2.10 (Common central value)**.** *Let $P, Q, R \in u\mathbb{C}[[u]]$, and let U, V, W be their zero-constant antiderivatives. The completed Jacobi algebra of $F = U(x) + V(y) + W(-x - y)$ is the complete algebra with a central generator w and relations*

$$P(x) = Q(y) = R(z) = w, \qquad x + y + z = 0.$$

Proof. The cyclic derivatives of $F = U(x) + V(y) + W(-x - y)$ are $\partial_x F = P(x) - R(-x - y)$ and $\partial_y F = Q(y) - R(-x - y)$. In the completed Jacobi algebra, setting both cyclic derivatives to zero equates $P(x)$ and $Q(y)$ to the common element $R(-x - y)$. This element commutes with x because it is equal to the series $P(x)$, and it commutes with y because it is equal to the series $Q(y)$. Because x and y generate the algebra topologically, this common value is central. Adjoining the generators $z = -x - y$ and $w = P(x)$ yields the displayed

presentation. Conversely, eliminating z and w via their defining relations defines the inverse homomorphism. Both homomorphisms are continuous with respect to the adic topologies defined by the augmentation ideals. □

Lemma 2.11 (One-variable preparation and its first jet)**.** *If $d \geq 2$ and $H(u) = h_d u^d + O(u^{d+1})$, $h_d \neq 0$, then $H(u) - w$ is a unit times a unique monic degree-d polynomial $M_H(u, w)$. Its lower coefficients lie in $w\mathbb{C}[[w]]$. After the shift $v = u + \rho_H(w)$, where $\rho_H = m_{d-1}/d$ and m_{d-1} is the coefficient of u^{d-1} in M_H, write*

$$M_H(v - \rho_H, w) = v^d - \sum_{i=0}^{d-2} T_{H,i}(w) v^i.$$

Then $T'_{H,0}(0) = 1/h_d$. If $\widetilde{H} - H \in u^M\mathbb{C}[[u]]$, $M \geq d+1$, then

$$\begin{aligned} \widetilde{\rho}_H - \rho_H &\in w^{\lfloor M/d \rfloor}\mathbb{C}[[w]], \\ \widetilde{T}_{H,i} - T_{H,i} &\in w^{\lceil (M-i)/d \rceil}\mathbb{C}[[w]]. \end{aligned} \tag{2.4}$$

Proof. By the formal Weierstrass preparation theorem in $\mathbb{C}[[w]][[u]]$, the series $H(u) - w$ factors as an invertible series times a unique monic distinguished polynomial $M_H(u, w) \in \mathbb{C}[[w]][u]$ of degree d. The roots give both the first jet and the continuity estimates. Set $w = q^d$. The d roots of the equation $H(u) = w$ admit Puiseux expansions $u_j = \zeta_d^j h_d^{-1/d} q + O(q^2)$ for $j = 0, \ldots, d-1$, where ζ_d is a primitive d-th root of unity. The product $\prod_{j=0}^{d-1}(u - u_j)$ is invariant under $q \mapsto \zeta_d q$, so it belongs to $\mathbb{C}[[w]][u]$. Uniqueness of the distinguished polynomial identifies it with $M_H(u, w)$. Its constant term is $-w/h_d + O(w^2)$. Shifting the variable by $\rho_H = m_{d-1}/d$ modifies this constant term only by terms in $O(w^2)$, which yields $T'_{H,0}(0) = 1/h_d$.

To establish the continuity bounds, consider a perturbation of H by an element of $u^M\mathbb{C}[[u]]$. We substitute $u = qv$ and divide the defining equation by q^d. The roots in the rescaled variable v are simple modulo q, while the perturbation has order at least q^{M-d}. Consequently, each root u_j changes by an amount of order $O(q^{M-d+1})$. In the elementary symmetric polynomial giving the coefficient of u^i, the remaining $d - i - 1$ root factors each contribute at least one power of q. The variation in the coefficient of u^i is therefore divisible by q^{M-i}. Returning to the base parameter $w = q^d$, this translates into the ceiling bound in (2.4). Setting $i = d - 1$ gives the corresponding floor bound for the shift ρ_H. Because ρ_H and all lower coefficients are divisible by w, the shift preserves these order bounds. A perturbation of the potential of order $dN + 1$ therefore alters the prepared coefficients only within $w^N\mathbb{C}[[w]]$, for any assignment of complex coefficients. □

Proof of Theorem 2.5. By Lemma 2.10, the completed Jacobi algebra of $U(x) + V(y) + W(-x - y)$ is presented by the three relations $P(x) - w = 0$, $Q(y) - w = 0$, and $R(z) - w = 0$ along with $x + y + z = 0$, where w is a central generator. We apply Lemma 2.11 to each of these three one-variable series and perform the corresponding shifts to eliminate their next-to-leading coefficients. In these centered variables, the linear relation becomes $\beta + \gamma + \delta = \rho_P + \rho_Q + \rho_R$, so we define the deformation parameter by $t_K = \ell(\rho_P + \rho_Q + \rho_R)$. The resulting polynomial coefficients match Karmazyn's marked quotient parameters [Karmazyn 19, Section 3.3]. Removing the invertible factors preserves the closed derivative ideal, and the inverse coordinate shifts recover the original variables.

Because the leading coefficients of P, Q, R are nonzero by assumption, Lemma 2.11 guarantees that $T'_{H,0}(0)$ is nonzero along each Dynkin arm. The first-jet condition of Lemma 2.7 is therefore satisfied. Proposition 2.8 constructs the associated crepant contraction with regular source, and Lemma 2.6 identifies its normalized basic contraction algebra with the completed Jacobi algebra. For a small contraction, the contraction algebra is finite over the target ring and supported at the closed point, which proves that it is finite-dimensional over $\mathbb{C}$. In marked length three, the divisorial alternative is determined by applying criterion (2.3) to the prepared coefficients. □

2.5 A nonzero quartic discriminant

Theorem 2.12 (Complete tails for $\Delta \neq 0$)**.** *Let $f = x^3 + O_4$ have reduced quartic $a_4(xy)^2 + b_4 xy^3 + c_4 y^4$. If $b_4^2 - 8a_4c_4 \neq 0$, then* Geo(Jac(f)). *The construction is a marked length-three E_6 arc. If $c_4 \neq 0$, it is small. If $c_4 = 0$, there is a cyclic right-equivalence to*

$$x^3 + xy^3 + p(y), \qquad p \in y^5\mathbb{C}[[y]],$$

and the construction is divisorial precisely for $p = 0$. All other members are finite-dimensional. The sole infinite-dimensional Jacobi algebra in this stratum, up to complete-algebra isomorphism, is Jac($x^3 + xy^3$)*, of growth dimension one.*

There are two normal forms, depending on whether the pure quartic coefficient vanishes. Their cyclic tangent calculations determine which higher-order terms can be absorbed by coordinate transformations and which surviving classes must be supplied by one-variable summands.

Lemma 2.13 (Two cyclic tangent calculations)**.** *Give x, y weights $3, 2$, respectively, and set $r_x = 3x^2 + y^3$, $S = xy^2 + yxy + y^2x$. Every cyclic homogeneous polynomial of weight $d > 9$ has the form*

$$[ur_x + vS] + \lambda[y^{d/2}], \qquad \mathrm{wt}(u) = d - 6, \quad \mathrm{wt}(v) = d - 7,$$

where the last summand occurs only for even d. For $\lambda_0 \neq 0$, the ordinary graded quotient

$$D_{\lambda_0} = \mathbb{C}\langle x, y\rangle/(x^2, S + \lambda_0 y^3, [x, y^3])$$

has cyclic quotient spanned by $1, x, xy$ and pure powers of y. In degrees at least seven the pure classes vanish unless the degree is divisible by three. If $L_b = y + bx$ and $t_b = 1 - \lambda_0 b$, then

$$[L_b^{3k}] = t_b^k[y^{3k}], \qquad [L_b^5] = \frac{1 + 5t_b^2}{6}[y^5]. \tag{2.5}$$

Proof. For the weighted calculation, we identify the cyclic classes remaining modulo the gradient terms r_x and S. For the ordinary graded calculation, the additional commutator relation $[x, y^3]$ will subsequently be removed by conjugating the generator while fixing the other generator.
(i) The quotient by r_x, S. In the quotient algebra defined by the relations $r_x = 0$ and $S = 0$, we set $T = x^2 = -y^3/3$. The element T commutes with both generators x and y. Over the central polynomial ring $\mathbb{C}[T]$, we apply the four reduction rules

$$xx \mapsto T, \qquad yyy \mapsto -3T, \qquad yyx \mapsto -xyy - yxy, \qquad yxyx \mapsto xyxy.$$

The fourth rule is obtained by comparing xS with Sx. Each rule decreases the word length or, at equal length, decreases the lexicographic order with $x < y$. Their left-hand sides produce ten proper overlaps and contain no inclusions. As confirmed by the exact check over $\mathbb{Z}[T]$ in Section 6, the two reductions at each overlap coincide. The irreducible words are

$$1, y, y^2, x, xy, xy^2, xyx, xyxy, xyxy^2, yx, yxy, yxy^2.$$

Confluence shows that these words form a free basis over $\mathbb{C}[T]$.

We now pass to the cyclic quotient. The relation $[Sy^j] = 0$ yields $3[xy^{j+2}] = 0$ for all $j \geq 0$, which implies $T[x] = T[xy] = 0$. The cyclic class of xyx is $T[y]$. Similarly, the identities $[Sx] = 0$ and $[Sxy] = 0$ give $[xyxy] = -2T[y^2]$ and $2[xyxy^2] = -T[y^3]$. These relations account for all twelve basis words and their central multiples. In weights strictly greater than nine, only pure powers of y can survive. Rotating each two-sided relation multiple to place the relation factor at the right produces the stated coefficient weights.
(ii) The quotient D_{λ_0}. In the algebra D_{λ_0}, we define $Z = y^3$ and $q = xyx$. Because $[x, y^3] = 0$, the element Z is central. Commuting x with the relation $S + \lambda_0 Z = 0$ yields $x(S + \lambda_0 Z) - (S + \lambda_0 Z)x = 0$. Modulo $x^2 = 0$, this simplifies to $xyxy = yxyx$, which makes q central with $q^2 = 0$. The same twelve words span D_{λ_0} as a module over $\mathbb{C}[Z]$, acting through the central element Z. Multiplying $S + \lambda_0 Z$ successively on the right by $1, y, y^2, x, yx$ and taking cyclic equivalence classes yields the five trace identities

$$[xy^2] = -\lambda_0[Z]/3, \qquad Z[x] = -\lambda_0 Z[y]/3, \qquad Z[xy] = -\lambda_0 Z[y^2]/3,$$
$$q[y] = (\lambda_0^2/3)Z[y], \qquad q[y^2] = (\lambda_0^2/6)Z[y^2].$$

These relations express all twelve cyclic classes in terms of $1, x, xy$ and pure powers of y. Multiplying the last two relations by q and using $q^2 = 0$ reveals that $Z^2[y]$ and $Z^2[y^2]$ vanish. In ordinary degrees seven and higher, the pure powers of y therefore vanish unless the degree is divisible by three.

To evaluate the powers of the linear form $L_b = y + bx$, we expand $L_b^3 = t_b Z + b^2 q$. Raising this expression to powers and using $[q] = 0$ yields the first formula in (2.5). Multiplying L_b^3 by L_b^2 and substituting the trace identities yields the second formula in degree five. In ordinary degree $n \geq 5$, every cyclic homogeneous polynomial can therefore be written as $[\mu y^n + ux^2 + v(S + \lambda_0 y^3) + k[x, y^3]]$, where u, v, k are homogeneous polynomials of degrees $n - 2, n - 3, n - 4$, respectively. □

Proof of Theorem 2.12. The classification divides according to whether the pure quartic coefficient c_4 vanishes. In each case, coordinate transformations absorb all terms lying in the cyclic tangent spaces, while the surviving cyclic classes are matched by the one-variable summands in the three-direction framework of Theorem 2.5. We then determine which of the resulting algebras are finite-dimensional and identify the unique divisorial member.

The branch $c_4 = 0$. When $c_4 = 0$, a shear followed by rescaling transforms the reduced quartic into the form xy^3. We then eliminate all cyclic xx terms by applying Lemma 2.2. With respect to the weights $\mathrm{wt}(x, y) = (3, 2)$, all terms beyond the cubic and quartic possess weight strictly greater than nine. We apply the first tangent calculation of Lemma 2.13 in order of increasing weight, retaining only the residual pure powers of y. In weight d, the coordinate adjustments $x \mapsto x - u$ and $y \mapsto y - v$ increase the generator weights by $d - 9 > 0$. Nonlinear

contributions produced by multiple replacements have weight at least $9 + 2(d-9) > d$, so each cancellation is preserved by subsequent steps. The infinite composition of these substitutions and their inverses converges in the weight-adic topology, bringing the potential to the normal form $x^3 + xy^3 + p(y)$.

To construct the geometric realization, set $\beta = y + x/2$ and $\gamma = -y + x/2$, and consider the three-direction potential

$$G_h = x^3 + \beta^4/4 - \gamma^4/4 + h(\beta), \qquad h \in u^5\mathbb{C}[[u]].$$

Setting $h = 0$ yields the cyclic equivalence class $x^3 + xy^3 + x^3y/4$, whose leading error term $x^3y/4$ has weight eleven. At each even weight $2m$, Lemma 2.13 shows that the residual classes are spanned by $[y^m]$. The leading weighted term of $\beta^m = (y + x/2)^m$ is y^m. Every term containing x has higher weight. We may therefore choose the coefficient of u^m in $h(u)$ to cancel the remaining class $[y^m]$. At odd weights, all cyclic terms lie in the gradient image, so no one-variable correction is required. Induction on weight simultaneously produces the series h and a convergent automorphism mapping G_h to the prescribed potential. In terms of the generators β and γ, the cubic term is $x^3 = (\beta + \gamma)^3 = -(-\beta - \gamma)^3$. The three one-variable derivatives therefore have exact orders $(3, 3, 2)$, so Theorem 2.5 supplies the geometric contraction.

The branch $c_4 \neq 0$. When $c_4 \neq 0$, we choose a scalar τ satisfying the quadratic relation $a_4 + b_4\tau + 2c_4\tau^2 = 0$. The shear $y \mapsto y + \tau x$ eliminates the alternating quartic term and satisfies $(b_4 + 4c_4\tau)^2 = \Delta \neq 0$. Rescaling coordinates and clearing newly created cyclic xx terms yields $f_c = x^3 + xy^3 + cy^4 + O_5$ with $c \neq 0$. We then choose a root r of the quadratic equation

$$4cr(1-r) + 2r - 1 = 0, \quad L_1 = y + rx, \quad L_2 = -y + (1-r)x, \quad a = c(1-r) + 1/4, \quad d = cr - 1/4.$$

This quadratic equation excludes the values $r = 0$ and $r = 1$, and determines the coefficients $a = (1-r)/(4r)$ and $d = -r/(4(1-r))$. The identities $a + d = c$, $4(ar - d(1-r)) = 1$, and $ar^2 + d(1-r)^2 = 0$ ensure that the linear combination $aL_1^4 + dL_2^4$ matches the target reduced quartic $xy^3 + cy^4$. All other terms of degree four are removed by a quadratic coordinate adjustment to x.

We then invoke the second calculation of Lemma 2.13 with $\lambda_0 = 4c$ and $t = 1 - 4cr = r/(1-r) \neq 0$. In degrees divisible by three, powers of L_1 supply the residual cyclic class in degree $3k$ with nonvanishing coefficient t^k. In degree five, L_1^5 and L_2^5 have cyclic classes with coefficients $(1+5t^2)/6$ and $-(1+5t^{-2})/6$. These coefficients cannot both vanish, because that would require $t^2 = -1/5$ and $t^2 = -5$ simultaneously. Suitable linear combinations in the one-variable series $h_1(L_1)$ and $h_2(L_2)$, beginning in degree five, therefore match every surviving cyclic class.

The relation multiples are removed degree by degree. Gradient adjustments eliminate the terms $[ux^2 + v(S + 4cy^3)]$. To remove the commutator multiple $[k[x, y^3]]$, we apply the exponential of the derivation defined by $D(x) = [k, x]$ and $D(y) = 0$. This exponential automorphism preserves the cyclic cubic x^3. Because the derivation increases word length by $n - 4$, the second variation of the quartic has degree $2n - 4 > n$, which enables an inductive cancellation. The coordinate transformations and their inverses converge for arbitrary complex coefficients, producing the three-direction potential

$$x^3 + aL_1^4 + dL_2^4 + h_1(L_1) + h_2(L_2).$$

Because $L_1 + L_2 = x$, this potential has derivative orders $(3, 3, 2)$. The derivatives of the prepared constant coefficients for the two quartic arms are $1/(4a)$ and $1/(4d)$. Their sum is $c/(4ad) \neq 0$. Consequently, the length-three divisorial condition $B_0 = -C_0$ fails at first order, which proves that the resulting contraction is small.

Dimension of the pure-series family. Let $A_p = \mathrm{Jac}(x^3 + xy^3 + p(y))$. The element $T = x^2 = -y^3/3$ is central. When $p = 0$, the twelve normal words of Lemma 2.13 form a free basis for A_0 over $\mathbb{C}[[T]]$ after completion. The weights of these basis words are bounded and the weight of T is six, so completing in the weight-adic topology amounts to completing the coefficients over $\mathbb{C}[T]$. The Jacobson radical modulo T is nilpotent. Because $T \in J^2$, the radical filtration and the T-adic filtration are cofinal with linear bounds on their degrees, which establishes that $\mathrm{Jdim}\, A_0 = 1$.

When $p \neq 0$, the initial relations with respect to the weights $(3, 2)$ remain $3x^2 + y^3$ and S. The associated graded algebra is therefore a quotient of the unperturbed graded algebra. Lifting the twelve spanning words and reducing successively in the complete algebra shows that A_p is generated by at most twelve elements as a module over $\mathbb{C}[[T]]$.

To prove finiteness, suppose that $p \neq 0$ and the generic algebra $A_p \otimes_{\mathbb{C}[[T]]} \mathbb{C}((T))$ is nonzero. A simple module over an algebraic closure of $\mathbb{C}((T))$ has dimension $r \leq 3$, because its image in the matrix algebra has dimension $r^2 \leq 12$. Write X, Y for the matrices of the generators. Setting $t^3 = T$, the eigenvalues of the diagonalizable matrix Y are $\lambda_i = \zeta_i t$, where $\zeta_i^3 = -3$. On the λ_i-eigenspace, the diagonal block of the second cyclic derivative relation reads $3\lambda_i^2 X_{ii} = -p'(\lambda_i)I$. Hence the trace $\mathrm{tr}\, X$ belongs to $\mathbb{C}((t))$. On the other hand, the relation $X^2 = t^3 I$ implies that $\mathrm{tr}\, X$ is an integer multiple of $t^{3/2}$. Comparing valuations forces $\mathrm{tr}\, X = 0$, so that the two eigenvalues of X have equal multiplicity and $r = 2$.

Now expand $p(y) = h_n y^n + \cdots$ with $h_n \neq 0$ and $n \geq 5$. The two contributions to the trace have leading sum proportional to $2\zeta_i^{n-3}$ for a repeated eigenvalue of Y, or to $\zeta_i^{n-3}(1+\omega^{n-3})$ for two distinct eigenvalues, where $\omega^3 = 1$. In both cases the leading sum is nonzero, contradicting the condition $\operatorname{tr} X = 0$. The generic algebra must therefore vanish. A sufficiently high power of T kills all twelve generators, which proves that A_p is finite-dimensional over $\mathbb{C}$.

A divisorial contraction in this marked family would possess a nonvanishing contraction algebra at the generic point of its exceptional image curve. On the split generic surface, a generator direct summand pushes forward to a nonfree maximal Cohen–Macaulay module for the contracted (-2)-curve, and its identity endomorphism persists in the stable endomorphism algebra. This rules out the divisorial alternative for the finite-dimensional members of this family. The growth-one calculation already excludes a small contraction for $p = 0$.

The divisorial member. We now realize the unperturbed algebra

$$\operatorname{Jac}(x^3 + xy^3).$$

We select the arc in the marked E_6 deformation space defined by

$$\beta^3 = w, \qquad \gamma^3 = -w, \qquad \delta^2 = -w/3, \qquad \beta + \gamma + \delta = 0.$$

The constant coefficients of all three polynomials have nonzero first derivatives at the origin, which guarantees that the source is regular. The two cubics have opposite sets of roots, producing persistent coefficient-two roots and yielding a divisor-to-curve contraction. The associated three-direction potential is $-\delta^3 + \beta^4/4 - \gamma^4/4$. Substituting $x = \beta + \gamma$ and $y = (\beta - \gamma)/2$ yields the cyclic equivalence class

$$x^3 + xy^3 + \tfrac{1}{4}x^3 y.$$

It remains to verify that eliminating the unwanted term $x^3 y/4$ introduces no pure series in y. Consider the parity involution ι defined by $\iota(x) = -x$ and $\iota(y) = y$. The leading potential $x^3 + xy^3$ and the discrepancy $x^3 y/4$ are both odd under ι. At each weight, we apply the odd projector $(1 - \iota)/2$ to the tangent decomposition provided by Lemma 2.13. Because r_x is even and S is odd under ι, this projector extracts an odd x-adjustment and an even y-adjustment, while completely eliminating the pure y-power summand, which is even. The resulting coordinate substitution commutes with ι, so that the error term in the next weight remains odd. The weight-adic limit gives the cyclic potential $x^3 + xy^3$ and identifies the completed Jacobi algebra of the arc.

For another divisor-to-curve model of the same algebra, see [Brown–Wemyss 26, Theorem 1.7]. □

2.6 The pure quartic and the two higher-jet conditions

When the reduced quartic is the pure fourth power y^4, we assign weights $\operatorname{wt}(x, y) = (4, 3)$ so that both leading terms x^3 and y^4 have weighted degree twelve. Among higher terms of weight below twenty-four, the surviving necklaces have weights $17, 20, 21$. Their coefficients govern the realization for marked lengths four and six.

Lemma 2.14 (The weighted y^4 reduction)**.** *Every $x^3 + y^4 + O_5$ is right-equivalent, with the entire tail retained, to*

$$x^3 + y^4 + \alpha_{17} xyxy^2 + \alpha_{20}(xy^2)^2 + \alpha_{21}(xy)^3 + O_{\mathrm{wt}}(24), \qquad \operatorname{wt}(x, y) = (4, 3),$$

where every remaining necklace avoids the cyclic subwords xx and yyy. The corresponding ordinary remainder has order at least seven.

Proof. In weighted degree $n > 12$, rotate each bad necklace to place a forbidden subword at the beginning. The sum of all such bad terms can then be expressed cyclically as $x^2 u_n$ plus $y^3 v_n$. The elementary substitutions $x \mapsto x - u_n/3$ and $y \mapsto y - v_n/4$ raise the weights of the generators by $n - 12 > 0$. They cancel these terms to first order and perturb the potential nonlinearly only in weights strictly greater than n. The substitutions and their inverses therefore converge in the weight-adic topology.

Because the formal tail begins in ordinary degree five, the steps at weights thirteen and fourteen are trivial. In particular, they introduce no $(xy)^2$ term. Every surviving higher necklace must avoid adjacent letters xx and subwords yyy, so each occurrence of x must be followed by either one or two letters y. Surviving necklaces are therefore concatenations of the two basic blocks xy of weight seven and xy^2 of weight ten, containing at least two occurrences of x. Listing all block words of weighted degree below twenty-four yields the three displayed terms of weights seventeen, twenty, and twenty-one. Every remaining higher necklace has weighted degree at least twenty-four and ordinary word length at least seven. □

Theorem 2.15 (Length-four realization)**.** *For every $a \neq 0$ and every cyclic formal tail $h \in O_6$,*

$$f = x^3 + y^4 + a\, xyxy^2 + h$$

is right-equivalent to $x^3 + V(y) + W(y - x)$, where $V(u) = u^4 + O(u^5)$, $W(u) = (a/5)u^5 + O(u^6)$. Its Jacobi algebra is finite-dimensional and geometric through a smooth marked length-four small contraction.

Theorem 2.16 (Nonzero sextic-pair realization)**.** *For $b, c \neq 0$, every*

$$f = x^3 + y^4 + b(xy^2)^2 + c(xy)^3 + O_7$$

is right-equivalent to $x^3 + V(y) + W(y - \lambda x)$, where

$$\lambda = -3c/(2b), \quad d_0 = b/(3\lambda^2), \quad V(u) = u^4 + O(u^5), \quad W(u) = d_0 u^6 + O(u^7).$$

It is realized by a smooth marked length-six small contraction and has finite-dimensional Jacobi algebra.

The proofs of Theorem 2.15 and Theorem 2.16 apply a common method to modify the one-variable series while controlling the compensating coordinate changes. The order bounds established in the following lemma allow these transfers to be iterated to all orders for an arbitrary formal tail.

Lemma 2.17 (Positive-order family transfer)**.** *Let $G = U(x) + V(y) + W(L)$, $L = y - \lambda x$, $\lambda \neq 0$. Suppose U' has exact order $d \geq 2$ and $W'(u) = bu^e + O(u^{e+1})$, $e > d$, $b \neq 0$. For $k \geq 1, s \geq 0$ with $r = d(k-1) + s \geq 2$ and for every $\rho \in \mathbb{C}$, there exist a modified U_ρ and an identity-tangent automorphism Ψ_ρ fixing y, such that*

$$\begin{aligned} \operatorname{ord}(U_\rho - U) &\geq dk + s, \\ \operatorname{ord}(\Psi_\rho(x) - x),\ \operatorname{ord}(\Psi_\rho^{-1}(x) - x) &\geq r, \\ [\Psi_\rho(U_\rho(x) + V(y) + W(L))] &= [G] + \rho b^k [L^{ek} x^s] + O_{ek+s+1}. \end{aligned}$$

The y-version uses V' of order d, fixes x, and adds $\rho b^k [L^{ek} y^s]$ in degree $ek + s$.

Proof. We vary the one-variable series U along a one-parameter flow and compensate with a coordinate substitution in x while holding y fixed. The noncommutative telescoping identity below ensures that the net variation of the potential becomes a multiple of a power of $W'(L)$.

Introduce a formal time parameter τ, set $U_0 = U$, and define the family of potentials $G_\tau = U_\tau(x) + V(y) + W(L)$. We set $A_\tau = U_\tau'(x)/\lambda$ and $B = W'(L)$, and consider the differential system

$$\begin{aligned} \partial_\tau U_\tau(u) &= \rho u^s (U_\tau'(u)/\lambda)^k, \\ C_\tau &= \sum_{i=0}^{k-1} B^{k-1-i} x^s A_\tau^i, \qquad D_\tau(x) = -\rho C_\tau/\lambda, \quad D_\tau(y) = 0. \end{aligned}$$

For a solution of this system, let Ψ_τ be the coordinate flow satisfying $\partial_\tau \Psi_\tau = \Psi_\tau D_\tau$ with initial condition $\Psi_0 = \mathrm{id}$. Noncommutative telescoping and cyclic rotation of words yield

$$[x^s A_\tau^k] - [B^k x^s] = [C_\tau (A_\tau - B)], \quad \partial_\tau [\Psi_\tau(G_\tau)] = \rho [\Psi_\tau(B^k x^s)].$$

The order estimates also give existence of this flow. Because $r = d(k-1) + s \geq 2$, each step in the coefficient recursion for $\partial_\tau U_\tau$ increases the word order by at least $r - 1 > 0$. The coefficients of U_τ are therefore polynomials in the parameter τ. The element C_τ has order at least r, so the flow equation $\partial_\tau \Psi_\tau = \Psi_\tau D_\tau$ with $\Psi_0 = \mathrm{id}$ can be solved order by order, yielding coefficients for Ψ_τ that are polynomial in τ. On each finite quotient $F_2/\mathfrak{m}_2^N$, the substitution acts unipotently. Its inverse is likewise polynomial in τ and satisfies the same order bound. We may therefore evaluate the flow at $\tau = 1$ compatibly on all finite quotients.

The degree-$ek + s$ term on the right-hand side of the flow equation equals $\rho b^k [L^{ek} x^s]$ identically throughout the flow. Integrating these polynomial coefficients from $\tau = 0$ to $\tau = 1$ yields the asserted leading term. For the y-branch, we set $A_\tau = -V_\tau'(y)$, $\partial_\tau V_\tau = \rho u^s (-V_\tau')^k$, $C_\tau = \sum_{i=0}^{k-1} B^{k-1-i} y^s A_\tau^i$, and $D_\tau(y) = \rho C_\tau$, while leaving x fixed. The exact flow identity then gives the corresponding leading variation and order estimates. □

Proof of Theorem 2.15. We choose the higher-order coefficients of the one-variable series V and W successively to match the potential, and then absorb the relation multiples through coordinate changes. The auxiliary quotient below identifies the specific cyclic classes that must be supplied by the three-direction family.

Residual cyclic classes. Write $u = xy$, $v = xy^2$, $g = xyxy^2$, and $q_{17} = [x, \partial_x g]$. To determine the degree-m terms that survive the principal gradient corrections and coordinate adjustments, consider the graded algebra

$$D_{17} = \mathbb{C}\langle x, y \rangle / (x^2, y^3, q_{17}).$$

Every non-forbidden mixed necklace is a cyclic word in the basic blocks u and v. For $m \geq 6$, we show that the residual cyclic classes are spanned by $[u^{m/2}]$ when m is even, together with $[u^{2k-2} v^2]$ when $m = 4k + 2$. All odd degrees in this range vanish in the cyclic quotient. Modulo the quadratic and cubic monomial relations $x^2 = 0$ and $y^3 = 0$, direct cyclic differentiation of g yields

$$q_{17} = uv + vu - yvx - y^2 ux.$$

Every nonempty word H in the blocks u and v begins with the letter x. Consequently, the cyclic classes $[Hyvx]$ and $[Hy^2ux]$ vanish because adjacent letters xx appear across the cyclic boundary. This leaves the cyclic relation $[Huv] = -[Hvu]$, which demonstrates that in any cyclic word containing at least three blocks, exchanging two adjacent unlike blocks changes the sign of the cyclic class. Modulo the monomial relations, we also have $q_{17}y^2 = -yv^2 - y^2uv$. Multiplying on the left first by y and then by x shows that $v^3 = 0$ in D_{17}. Sorting the blocks into the standard form u^iv^j therefore kills any cyclic class containing three or more occurrences of the block v.

For a sorted mixed word u^iv^j with at least three blocks, cyclically rotating one block u from the beginning to the end and commuting it back past the j factors of v introduces the sign $(-1)^j$. Moving a factor of v in the same cyclic manner introduces $(-1)^i$. Hence both exponents i and j must be even for the cyclic class to be nonzero. The only mixed two-block word is uv, which has degree five. This establishes the asserted list of residual classes in every degree $m \geq 6$. Rotating the two-sided relation multiples expresses the remaining discrepancy in the form $[Hq_{17} + x^2u_m + y^3v_m]$, where the coefficient polynomials have degrees $m-5, m-2, m-3$, respectively.

Matching degree six. In the seed potential $x^3 + y^4 + (a/5)(y-x)^5$, the unique non-forbidden fifth necklace has coefficient a. We remove all forbidden fifth necklaces using principal corrections in x and y. Set $L = y - x$. To adjust the sixth jet, we apply the y-flow of Lemma 2.17 prior to those normalizing substitutions, using parameters $(d, e, k, s) = (3, 4, 1, 2)$ and amplitude $-A/4$. The leading variations are Ay^5 in the series V and $-Ay^2/4$ in the coordinate y. By the exact flow identity, their net variation in degree six is $-aA[L^4y^2]/4$. Because the normalizing substitutions are tangent to the identity, they preserve this leading term.

Expanding L^4y^2 in terms of blocks, the only necklace that avoids forbidden subwords cyclically is $xy^2xy^2 = v^2$. For the second independent direction, we add $B\xi^6$ to the one-variable series $W(\xi)$. This adds $B[L^6]$ to the potential, which expands to three cyclic rotations of v^2 and two cyclic rotations of u^3 with opposite signs. Combining these two contributions yields

$$(-aA/4 + 3B)v^2 - 2Bu^3 \tag{2.6}$$

in the cyclic quotient. The linear transformation sending (A, B) to the coefficients of v^2 and u^3 has determinant $a/2$. Since $a \neq 0$, we can match both residual coefficients in degree six.

Higher degrees and convergence. Assume that the current family agrees with the prescribed potential below degree $m \geq 7$. At degree $m = 4k$, adding $\mu\xi^m$ to $W(\xi)$ contributes $2\mu[u^{2k}]$. At degree $m = 4k+2$, the same variation produces the coefficient -2μ on $[u^{2k+1}]$. The second independent direction is provided by the y-flow with exponent k and shift $s = 2$, whose leading contribution is $\rho a^k[L^{4k}y^2]$.

The appended factor y^2 forces the final block of the word to be v after cyclic rotation. For a necklace containing exactly two factors of v, the other pair of consecutive y's may appear in any of the first $2k-1$ positions. Commuting the two factors of v together produces alternating signs that sum to one. Necklaces containing three or more factors of v vanish after sorting, by $v^3 = 0$, while the presence of the appended pair excludes the pure u-class. The flow therefore contributes $\rho a^k[u^{2k-2}v^2]$ to the cyclic quotient. In odd degrees, there are no surviving cyclic classes to match. At each step, we apply the transfer flow to the current family of potentials. The family update has order at least $3k+2$, while the coordinate map and its inverse differ from the identity by terms of order at least $3k-1$.

To eliminate the relation multiple $[Hq_{17}]$, we apply the continuous automorphism that conjugates the generator x by $\exp(H/a)$ while holding y fixed. This coordinate substitution preserves the cyclic cubic x^3 and leaves the pure quartic y^4 invariant. The first variation of ag under this substitution is $[Hq_{17}]$, while the second variation has degree $2m-5 > m$. Principal gradient corrections in x and y remove the multiples of x^2 and y^3, with signs chosen to cancel the remaining error terms. The updates to the one-variable series, the coordinate transformations, and their inverses all stabilize on every finite word quotient $F_2/\mathfrak{m}_2^N$. Passing to the inverse limit and setting $X = x$ and $Y = y - x$ yields a three-direction potential with derivative orders $(2, 4, 3)$. Theorem 2.5 therefore establishes the claimed smooth marked length-four small contraction. □

Proof of Theorem 2.16. We use the same division into residual cyclic classes and relation multiples as in the length-four proof. In this setting, the existence of a central element of degree five reduces the cyclic calculation to a finite-dimensional auxiliary algebra.

(i) A finite quotient. Set $L = y - \lambda x$, $g_6 = b(xy^2)^2 + c(xy)^3$, $q_6 = [x, \partial_x g_6]$, and $D_6 = \mathbb{C}\langle x, y\rangle/(x^2, y^3, q_6)$. To determine the cyclic classes that must be matched in each degree, we first identify a central element of degree five. In the quotient algebra by the ideal (x^2, y^3), the linear operation $h \mapsto [x, \partial_x h]$ kills every bad necklace containing forbidden subwords. For a necklace x^2a, cyclic differentiation gives $[x, xa + ax] = [x^2, a]$, while all other terms retain an explicit factor of x^2. Similarly, any representative y^3a retains y^3 under every cyclic x-derivative. The non-forbidden coefficients of d_0L^6 are $3d_0\lambda^2 = b$ and $-2d_0\lambda^3 = c$. Consequently, we have the congruence

$$q_6 = -6\lambda d_0[x, L^5] \pmod{(x^2, y^3)}.$$

Because $[y, L^5] = \lambda[x, L^5]$, the element $z = L^5$ commutes with both generators and is therefore central in D_6.

After rescaling the coordinate x by λ, the quotient algebra by the ideal generated by z becomes

$$B_6 = \mathbb{C}\langle x, y\rangle/(x^2, y^3, (y-x)^5).$$

To analyze this graded algebra, we index the columns in degree n by words avoiding adjacent letters xx and subwords yyy. For each pair of non-forbidden words a, b with $|a| + |b| = n - 5$, we expand the product $a(y-x)^5 b$ and discard all words containing forbidden subwords. The resulting rows span the relations in degree n. For the cyclic cocenter $HH_0(B_6)$, we identify cyclic permutations of words and set any class with a forbidden cyclic rotation to zero. The rows $[(y-x)^5 b]$ then suffice to span the relations, because a left multiplier rotates cyclically to the right. Gaussian elimination over the field of rational numbers gives the vector space dimensions in degrees zero through fifteen

$$\dim(B_6)_n : 1, 2, 3, 4, 5, 6, 6, 6, 6, 6, 5, 4, 3, 2, 1, 0,$$
$$\dim HH_0(B_6)_n : 1, 2, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.$$

Adding the rows corresponding to the seven elements $1, y, y^2, L, L^2, L^3, L^4$ in their respective degrees yields full column rank. The computer algebra verification in Section 6 confirms these matrix ranks. Because the degree-fifteen component of the algebra B_6 vanishes identically, every word of degree fifteen and higher vanishes as well.

(ii) Lifting the cyclic generators. Because z is central in D_6, we have the isomorphism $HH_0(D_6)/zHH_0(D_6) = HH_0(B_6)$. For any homogeneous cyclic class in D_6, we subtract a linear combination of the seven lifts of the spanning classes of $HH_0(B_6)$. The resulting remainder belongs to $zHH_0(D_6)$, and can therefore be written as z times a cyclic class of degree lower by five. By induction on degree, these seven elements generate the cyclic cocenter $HH_0(D_6)$ as a module over the polynomial ring $\mathbb{C}[z]$.

Consequently, in every degree $m \geq 7$, the residual cyclic classes are spanned by $[L^m]$ and, when $m = 5k + j$ with $j \in \{1, 2\}$, by $[L^{5k} y^j]$. After these surviving classes have been matched, the remaining discrepancy in degree m takes the form $[x^2 u_m + y^3 v_m + H_m q_6]$, where the coefficient polynomials have degrees $m-2, m-3, m-6$, respectively.

(iii) Matching the formal tail. The seed potential $x^3 + y^4 + d_0 L^6$ has the required sextic coefficients $3d_0\lambda^2$ and $-2d_0\lambda^3$. We remove its forbidden sixth-degree necklaces using principal corrections in x and y. Assume by induction that the potential matches the prescribed series through degree $m-1$. Adjusting the degree-m coefficient in the one-variable series W supplies the required multiple of $[L^m]$. To supply the class $[L^{5k} y^j]$, we apply the second transfer flow of Lemma 2.17 with parameters $d = 3, e = 5, s = j$, setting the flow amplitude ρ equal to the target coefficient divided by $(6d_0)^k$.

The order condition $r = 3(k-1) + j$ is at least two in all relevant degrees. The minimal value of k is two when $j = 1$ (degree eleven) and one when $j = 2$ (degree seven). The exact flow identity of Lemma 2.17 therefore applies from the very first step, modifying the series V only in orders at least $3k + j$.

Finally, we remove the relation multiple $[H_m q_6]$ by applying the continuous automorphism that conjugates the generator x by $\exp(H_m)$ while holding y fixed. The resulting first variation provides $[H_m q_6]$, while the second variation on g_6 has degree $2m - 6 > m$. Principal gradient corrections in x and y remove the other relation multiples. Each previously matched degree is preserved throughout these adjustments. As m tends to infinity, the orders of the family updates and of the differences of the coordinate maps and inverse maps from the identity tend to infinity. Their limits give a continuous right-equivalence in the word-adic topology. Setting $X = \lambda x$ and $Y = -y$ yields derivative orders $(2, 3, 5)$. Theorem 2.5 applies to these orders and constructs the claimed smooth marked length-six small contraction. □

3 Realization of the length-five stratum

For the alternating quartic, the growth bound $\mathrm{Jdim} \leq 1$ forces the pure fifth coefficient to be nonzero. We first normalize this coefficient, and then prove that every Jacobi algebra with the resulting five-jet is finite-dimensional. The geometric realization follows by polynomial truncation and gluing.

3.1 The alternating quartic and its fifth jet

Theorem 3.1 (A uniform weighted growth obstruction)**.** *If every word of f has weight at least six for* $\mathrm{wt}(x, y) = (2, 1)$*, then for every $N \geq 1$*

$$\dim_{\mathbb{C}}\big(\mathrm{Jac}(f)/J^N\big) \geq \#\{(a, b, c) \in \mathbb{Z}^3_{\geq 0} : 2a + b + 3c < N\} \geq \big(\lfloor (N-1)/6 \rfloor + 1\big)^3. \tag{3.1}$$

In particular $\mathrm{Jdim}\,\mathrm{Jac}(f) \geq 3$.

Proof. We first count the normal words in a formal deformation of a fixed homogeneous potential. Specialization of the finite relation matrix then establishes the dimension lower bound at every point in the coefficient space.

Let K_N denote the closed subspace spanned by words of weight at least N. Because cyclic differentiation reduces $(2,1)$-weight by at most two, only coefficients of the potential in weights six through $N+1$ can affect the truncated quotient. We vary these coefficients independently about the seed potential $f_0 = (xy-yx)^2/2$, introducing a formal perturbation $g = \sum e_i h_i$ over the power series ring $T = \mathbb{C}[[e_1, \dots, e_r]]$. We freely adjoin a generator ζ of weight three. The cyclic derivatives of the perturbed potential $\zeta(xy-yx) - \zeta^2/2 + g$ produce the reduction rules

$$yx \longmapsto xy - \zeta, \quad \zeta y \longmapsto y\zeta + \partial_x g, \quad \zeta x \longmapsto x\zeta - \partial_y g.$$

We truncate the relations first in the parameter degree of the coefficients e_i. Each cyclic derivative correction increases the parameter degree. At any fixed parameter degree, the remaining terms strictly decrease the word length or, at equal length, reduce the number of inversions with respect to the ordering $x < y < \zeta$. This guarantees termination, and none of the reduction rules decreases the weighted degree. The unique overlap among the leading words is ζyx. The two distinct reduction paths for this overlap differ by $-[y, \partial_y g] - [x, \partial_x g]$, which vanishes identically by the cyclic commutator identity $[x, \partial_x g] + [y, \partial_y g] = 0$. Disjoint reductions commute.

The terminating confluent reductions show that the ordered monomials $x^a y^b \zeta^c$ satisfying $2a + b + 3c < N$ yield a unique normal form at every parameter cutoff. Passing to the inverse limit in parameter degree shows that the truncated algebra is a finite free T-module with these ordered monomials as a basis. Eliminating the auxiliary generator $\zeta = xy - yx$ identifies this module with the weighted quotient of the deformed completed Jacobi algebra.

We now extend this dimension bound from the formal neighborhood of the seed potential to arbitrary complex coefficients. Consider the finite relation matrix defining the truncated word quotient. Its entries are polynomials in the coefficients e_i. The generic rank of this matrix over the quotient field of T equals the total dimension of the word space minus the number of ordered monomials. Every minor of size strictly greater than this generic rank vanishes identically as a formal power series at the origin, and because its entries are polynomials in the coefficients, it must vanish identically as a polynomial everywhere. Specialization to any complex coefficient values can only decrease the rank of the relation matrix, which can only increase the dimension of the resulting quotient. Translating the coefficient coordinates by the seed potential f_0 therefore preserves the same dimension lower bound for every potential of weight at least six. At any fixed weight cutoff, the closed derivative ideal and the ordinary derivative ideal have identical images on the finite-dimensional word space, so the higher-order formal tail introduces no additional relations at that cutoff.

Because the powers of the Jacobson radical satisfy $J^N \subseteq K_N \operatorname{Jac}(f)$, the radical quotient surjects canonically onto the weighted quotient. This establishes the first inequality in (3.1). For the second inequality, restricting each of the three nonnegative integer exponents a, b, c to the range $0 \le a, b, c \le \lfloor (N-1)/6 \rfloor$ yields the weight bound $2a + b + 3c \le 6\lfloor (N-1)/6 \rfloor < N$. The number of such triples is at least $(\lfloor (N-1)/6 \rfloor + 1)^3$, which completes the proof. □

Corollary 3.2 (The alternating-quartic five-jet)**.** *If $f = x^3/3 + (xy)^2/2 + O_5$ and $\operatorname{Jdim} \operatorname{Jac}(f) \le 1$, its pure y^5 coefficient is nonzero. A formal coordinate change and a nonzero scalar multiple put its five-jet into*

$$x^3/3 + (xy)^2/2 - xy^4 - y^5/5.$$

Proof. The only cyclic necklaces of degree five avoiding adjacent letters xx are $xyxy^2$, xy^4, and y^5. If the coefficient of y^5 were zero, every surviving word in the potential would have $(2,1)$-weight at least six, which by Theorem 3.1 would force $\operatorname{Jdim} \operatorname{Jac}(f) \ge 3$, contradicting the hypothesis $\operatorname{Jdim} \le 1$.

Let A, B, C denote the coefficients of $xyxy^2$, xy^4, and y^5, respectively, with $C \neq 0$. Rescaling coordinates by $x \mapsto s^2 x$ and $y \mapsto sy$, and multiplying the overall potential by s^{-6} with $s = -5C$, normalizes the leading pure power coefficient to $C = -1/5$. We then apply the shear $y \mapsto y + dx$ and restore the standard four-jet via the quadratic substitution $x \mapsto x - dxy - d^2x^2/2$. This shear shifts the coefficient B of the necklace xy^4 to $B - d$, while the restoring substitution affects only necklaces containing at least two occurrences of x. Choosing $d = B + 1$ normalizes this coefficient to -1. We then eliminate the bad fifth necklaces using a cubic coordinate correction in x, and finally apply the quadratic transformation $y \mapsto y - A'y^2$ to cancel the residual $A'xyxy^2$ term. All remaining terms have ordinary word length at least six. □

3.2 The length-five finiteness theorem

For the finiteness calculation, we work with generators of weights one and two. Proposition 3.8 relates these generators to the preceding five-jet through an explicit change of coordinates. Let

$$p(x) = a_4 x^4 + O(x^5), \qquad a_4 \neq 0, \qquad I_p' = p, \quad I_p(0) = 0,$$

and put

$$F_p = \frac{y^3}{3} + y^2x^2 + \frac{yxyx}{2} + yx^4 + \frac{x^6}{6} - I_p(x).$$

Theorem 3.3. *For every cyclic formal series $h \in O_6$, the algebra*

$$A = \mathrm{Jac}(F_p + h)$$

is finite-dimensional over $\mathbb{C}$.

If the completed Jacobi algebra were infinite-dimensional, its associated graded algebra would be infinite-dimensional as well. In that case, the associated graded algebra would appear as a quotient of an auxiliary graded algebra having a central simple generic fiber of dimension 25 over the parameter field, and this quotient would necessarily inherit the same generic rank 25. The fourth-order relation would then force $4 \mid 25$. To establish this divisibility, we let a lift of the graded central element act on the left. Right multiplication by elements of the algebra is linear with respect to this left parameter action, even when the lifted element itself is not central.

3.3 The auxiliary graded algebra

Assign weights $1, 2$ to x, y, respectively, and define

$$Q = y^2 + yx^2 + x^2y + xyx + x^4, \qquad S = yxy + y^2x + yx^3, \qquad B = \mathbb{C}\langle x, y\rangle/(x^4, Q).$$

The free polynomial identities

$$[S, y] = y[x, Q], \qquad [S, x] = [y, Q] + x[x, Q]$$

demonstrate that S commutes with both generators modulo $Q = 0$. Consequently, S is central in the quotient algebra B and has weighted degree five.

Lemma 3.4. *The algebra B is finite over $\mathbb{C}[S]$, and*

$$\dim_{\mathbb{C}} B/(S) = 40, \qquad B/(S-1) \cong M_5(\mathbb{C}).$$

Consequently $B \otimes_{\mathbb{C}[S]} \mathbb{C}(S)$ is central simple of dimension 25.

Proof. We complete the defining relations of B using the weighted degree-lexicographic term order. The script `nr_B5_generic_certificate.py` performs this completion using exact rational arithmetic, resolving every overlap and inclusion. Setting $S = 0$ yields 40 irreducible words, while setting $S = 1$ yields 25 irreducible words. Both calculations take place in the ordinary polynomial algebra.

For the specialization at $S = 1$, the 25-dimensional regular trace pairing has nonvanishing determinant

$$298023223876953125 = 5^{25},$$

and the center of the specialized algebra has dimension one. In characteristic zero, nondegeneracy of the trace pairing implies that the algebra is semisimple. Because the center is one-dimensional, there is a unique simple factor, which is isomorphic to the matrix algebra $M_5(\mathbb{C})$. The basis elements, irreducible words, and exact outputs are documented in `nr_B5_generic_certificate.json`.

Lifting the homogeneous basis words of $B/(S)$ to B, induction on weighted degree shows that these lifts generate B as a module over the polynomial ring $\mathbb{C}[S]$. To determine the generic fiber, adjoin an element z satisfying $z^5 = S$ to obtain the field extension $\mathbb{C}(S) \subset \mathbb{C}(z)$. Rescaling the generators by $x = zX$ and $y = z^2Y$ identifies $B \otimes_{\mathbb{C}[S]} \mathbb{C}(z)$ with the scalar extension of $B/(S-1)$ to the field $\mathbb{C}(z)$. Central simplicity and dimension 25 descend along this field extension. □

3.4 Filtered modules with a noncentral parameter

Let A be a separated complete algebra with a decreasing positive integer filtration F^nA, $F^0A = A$, finite-dimensional associated graded pieces, and $F^iAF^jA \subseteq F^{i+j}A$.

Lemma 3.5. *Suppose $\mathrm{gr}\, A$ is finite over the polynomial ring $\mathbb{C}[\sigma]$, where σ is central and homogeneous of degree $e > 0$. Choose a lift $s \in F^eA$ of σ. Regard A as a left $D = \mathbb{C}[[t]]$-module by $t \cdot a = sa$. Then A is a finite D-module and*

$$\mathrm{rank}_D A = \mathrm{rank}_{\mathbb{C}[\sigma]} \mathrm{gr}\, A.$$

Its D-torsion submodule is a right ideal and is finite-dimensional over $\mathbb{C}$.

Proof. Lift a finite homogeneous generating family of the graded module $\operatorname{gr} A$ over $\mathbb{C}[\sigma]$ to elements of A. Express any element of A by subtracting left powers of s times these lifts to eliminate the leading graded term, and repeat this division on the remainder. Because the filtration degrees strictly increase at each step, the coefficients of the lifts converge in the complete ring D. This establishes that A is finitely generated as a left D-module. If c denotes the maximum filtration degree among the chosen generators, this division process yields the containment bounds

$$F^{en+c}A \subseteq s^n A \subseteq F^{en}A.$$

The division process expresses an element of $F^{en+c}A$ using left powers of s with exponent at least n. A finitely generated graded $\mathbb{C}[\sigma]$-module of rank r has cumulative vector space dimension $rN/e + O(1)$ in degrees strictly below N, as follows from choosing a homogeneous basis modulo torsion. The two filtration inclusions therefore imply $\dim_{\mathbb{C}} A/s^n A = rn + O(1)$. By the structure theorem for finitely generated modules over the discrete valuation ring D, the leading coefficient of n in the dimension growth coincides with $\operatorname{rank}_D A$.

By associativity of the algebra multiplication, right multiplication by any element of A commutes with the left action of s. Consequently, the D-torsion submodule of A is preserved by all right multiplications, making it a right ideal. Being a finitely generated torsion module over the discrete valuation ring D, this submodule has finite length, and is therefore finite-dimensional over $\mathbb{C}$. □

Remark 3.6. Left multiplication by s supplies the D-module structure, while right multiplication supplies the D-linear determinant operators.

Lemma 3.7 (Filtered determinant test)**.** *In the situation of Lemma 3.5, assume the common rank is $r > 0$. Suppose $z \in F^1 A$, $m \geq 1$, $c \in \mathbb{C}^\times$, and a one-variable formal series v satisfy*

$$q(z) = cz^m(1 + zv(z)) = s + \epsilon, \qquad \epsilon \in F^{e+1}A.$$

Then m divides r.

Proof. The defining relation produces two evaluations for the valuation of the determinant of right multiplication by $q(z)$. Its leading filtered term yields r, whereas its factorization through z^m yields an integer multiple of m.
(i) Right multiplication on a free module. Let $T \subset A$ denote the finite-dimensional D-torsion submodule. The quotient module $M = A/T$ is then a free module of rank r over the discrete valuation ring D. We equip M with the quotient filtration $F^n M = (F^n A + T)/T$. Because the torsion submodule T is a right ideal, right multiplication $R_a : m \mapsto ma$ by any element $a \in A$ descends to a well-defined endomorphism on M, which is D-linear with respect to the left D-action. Centrality of the initial form of s in the associated graded algebra $\operatorname{gr} A$ implies that the commutator $[s, a] = sa - as$ belongs to filtration degree strictly greater than $e + \deg(a)$. This yields the inclusion

$$(R_s - t\,\mathrm{id})(F^n M) \subseteq F^{n+e+1}M.$$

Consequently, right multiplication by $q(z)$ takes the form $R_{q(z)} = t\,\mathrm{id} + E$, where the error operator E shifts the filtration by at least $e + 1$.
(ii) A lattice for the filtration. Adjoin an element u satisfying $t = u^e$ and work over the Laurent field $\mathbb{C}((u))$. To translate the filtration shifts into valuations, consider the lattice

$$\mathcal{L} = \sum_{n \geq 0} u^{-n}\big(F^n M \otimes_D \mathbb{C}[[u]]\big) \subset M \otimes_D \mathbb{C}((u)).$$

Set $M_u = M \otimes_D \mathbb{C}[[u]]$, and let c_0 be the largest weight of the homogeneous generators used in the proof of Lemma 3.5. The filtration comparison in that proof gives $F^n M \subset t^{\lfloor (n-c_0)/e \rfloor} M$ for $n \geq c_0$. Every summand in the definition of $\mathcal{L}$ is therefore contained in $u^{-c_0-e}M_u$, while the summand at $n = 0$ is M_u. The submodule $\mathcal{L}$ is therefore a full $\mathbb{C}[[u]]$-lattice in $M \otimes_D \mathbb{C}((u))$.
(iii) The two determinant valuations. On this lattice, any operator that increases filtration degree by j maps $\mathcal{L}$ into $u^j\mathcal{L}$. In particular, the error operator satisfies $E\mathcal{L} \subset u^{e+1}\mathcal{L}$, while t acts as scalar multiplication by u^e. It follows that

$$R_{q(z)} = u^e(I + uC) \quad \text{on } \mathcal{L}, \qquad v_t(\det R_{q(z)}) = r.$$

Indeed, the determinant of $R_{q(z)}$ has u-valuation er, and the valuations satisfy $v_u = e\, v_t$ on $\mathbb{C}((t))$. Because $z \in F^1 A$, the right multiplication operator R_z maps $\mathcal{L}$ into $u\mathcal{L}$. The operator $R_{1+zv(z)} = \mathrm{id} + R_z v(R_z)$ therefore reduces to the identity modulo u, which gives it unit determinant in $\mathbb{C}[[u]]^\times$. Taking determinants in the factorization $q(z) = cz^m(1 + zv(z))$ gives

$$r = m\, v_t(\det R_z).$$

Because the determinant $\det R_z$ is a nonzero element of $\mathbb{C}((t))$, its valuation $v_t(\det R_z)$ is an integer. Thus m divides r. □

3.5 The rank contradiction

Proof of Theorem 3.3. Decompose the formal tail as $h = g(x) + h_{\mathrm{mix}}$, where $g \in x^6\mathbb{C}[[x]]$ and every word in h_{mix} contains at least one occurrence of y. Any mixed word of ordinary word length at least six has $(1,2)$-weighted degree at least seven. We absorb the pure part of the tail into $\widetilde{p} = p - g'$, whose leading term remains a_4x^4. Direct cyclic differentiation of the potential F_p yields

$$\partial_y F_p = Q, \qquad \partial_x F_p = S + xQ - p(x).$$

Setting the cyclic derivatives of $F_p + h$ to zero in the completed Jacobi algebra A, the two defining relations become

$$Q \in F^5A, \qquad \widetilde{p}(x) = S + \big(\partial_x h_{\mathrm{mix}} - x\partial_y h_{\mathrm{mix}}\big),$$

where the parenthesized discrepancy has weighted degree at least six. The leading terms of these two relations with respect to the $(1,2)$-grading are x^4 and Q. There is therefore a canonical surjection of graded algebras

$$B \twoheadrightarrow \operatorname{gr} A.$$

By Lemma 3.4, the associated graded algebra $\operatorname{gr} A$ is a finitely generated module over the image of the central parameter ring $\mathbb{C}[S]$.

If A were infinite-dimensional over $\mathbb{C}$, its associated graded algebra $\operatorname{gr} A$ would be infinite-dimensional as well. As a finitely generated module over $\mathbb{C}[S]$, it would then necessarily have a nonzero generic fiber, because a finitely generated torsion module over a one-variable polynomial ring is finite-dimensional. This generic fiber is a nonzero quotient of the central simple algebra $B \otimes_{\mathbb{C}[S]} \mathbb{C}(S)$ of dimension 25 from Lemma 3.4. Because a central simple algebra admits no proper nonzero two-sided ideals, this quotient map is an isomorphism, which forces the generic rank of $\operatorname{gr} A$ over $\mathbb{C}[S]$ to be exactly 25.

We now apply Lemma 3.5 to the lift $s = S(x,y) \in F^5A$ of the central element S. Lemma 3.7 then applies with parameters $e = 5$, $z = x$, $q = \widetilde{p}$, and leading exponent $m = 4$. This forces 4 to divide 25, which is impossible. We conclude that A is finite-dimensional over $\mathbb{C}$. □

3.6 Polynomial truncation and length-five realization

Proposition 3.8 (Length-five realization)**.** *Every potential in Theorem 3.3 has a geometric Jacobi algebra, realized by a smooth length-five cE_8 flop. Equivalently, the alternating-quartic stratum with nonzero pure fifth coefficient is realized, including all formal tails.*

Theorem 3.3 establishes that the Jacobson radical is nilpotent. The following lemma translates this nilpotence exponent into an effective polynomial truncation bound.

Lemma 3.9. *If* $\operatorname{Jac}(f)$ *has radical nilpotence exponent L, then replacing f by its ordinary truncation through any degree $N \geq L+1$ leaves its closed Jacobi ideal unchanged.*

Proof. Let I and I_N denote the closed two-sided ideals generated by the cyclic derivatives of f and its degree-N polynomial truncation $f_{\leq N}$, respectively. Their generators agree modulo the maximal ideal power $\mathfrak{m}^N$, and by assumption $\mathfrak{m}^L \subseteq I$. Because $N \geq L+1$, we have the containment

$$\mathfrak{m}^L \subseteq I_N + \mathfrak{m}^{L+1}.$$

Multiplying this containment successively by powers of $\mathfrak{m}$ and substituting back recursively yields $\mathfrak{m}^L \subseteq I_N + \mathfrak{m}^{L+k}$ for arbitrarily large integers k. Because the ideal I_N is closed in the $\mathfrak{m}$-adic topology, taking the intersection over all k yields $\mathfrak{m}^L \subseteq I_N$. The differences between the cyclic derivatives of f and those of $f_{\leq N}$ belong to $\mathfrak{m}^N \subseteq \mathfrak{m}^L \subseteq I \cap I_N$, which proves the equality of derivative ideals $I = I_N$. □

Proof of Proposition 3.8. Applying the coordinate change $X = y + x^2$ and $Y = x$ transforms the potential into a series with five-jet

$$\frac{X^3}{3} + \frac{(XY)^2}{2} - XY^4 - \frac{a_4}{5}Y^5.$$

At the level of five-jets, we perform a shear $Y \mapsto Y + dX$. We restore the standard quartic form using a quadratic correction in X and clear newly introduced cyclic XX terms. This transformation alters the coefficient B of the necklace XY^4 to $B + 5Cd$, where $C \neq 0$ denotes the coefficient of the pure fifth power Y^5. We choose d to cancel this term completely. A subsequent quadratic correction in Y removes the remaining $XYXY^2$ term. Rescaling the generators and multiplying the potential by a nonzero scalar yields $X^3 + (XY)^2 + Y^5 + O_6$, where all higher-order terms remain in O_6.

By Theorem 3.3, the Jacobson radical of the Jacobi algebra has a finite nilpotence exponent L. We truncate the potential through a degree N satisfying $N \geq \max(5, L+1)$. By Lemma 3.9, this polynomial truncation preserves the closed Jacobi ideal, while the degree bound preserves the normalized five-jet.

We then apply Wemyss's gluing theorem [Wemyss 26, Sections 1.5–1.6, Eq. (1.3) and Theorem 1.7]. The fifth family listed in Theorem 1.6 of that paper is $X^3 + (XY)^2 + Y^5 + O_6$. The realization statement following Eq. (1.3) identifies the Jacobi algebra of a listed polynomial potential with the noncommutative deformation algebra of the associated two-chart curve neighborhood. Theorem 1.7 of [Wemyss 26] proves that contracting the central zero section produces a projective crepant contraction from a regular threefold source, forming a smooth length-five flop. Localizing and completing at the closed image of the curve and pulling back the regular source produces the required complete local contraction of type cE_8. □

Remark 3.10. For an explicit instance of the gluing construction, consider the two-series subfamily $G = F_p + g(x) + \ell(x)y$ with $g \in x^6\mathbb{C}[[x]]$ and $\ell \in x^5\mathbb{C}[[x]]$, and set $\widetilde{p} = p - g'$. In this setting, the free-necklace construction produces explicit transition functions between two affine charts. With formal series coefficients, these coordinate transitions are interpreted on the completed charts by

$$b = a^{-1}, \quad U = a^{-1}u, \quad V = a^3v + a^{-1}u^2 + u^3 + au^4 + a^2(u^5 - \widetilde{p}(u)) + a\ell(u).$$

This coordinate transition has Jacobian determinant -1. Its zero section is a curve $\mathbb{P}^1$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(-3)$. By [Brown–Wemyss 24, Corollary 6.4], the deformation algebra of this curve neighborhood is isomorphic to $\mathrm{Jac}(G)$. Truncating the series yields polynomial gluing data for this subfamily.

Remark 3.11. The subsequent classification of Brown and Wemyss provides an alternative deduction of the geometric realization step. In cyclic necklace notation, let $\mathcal{N}_{2,2} = X^2Y^2 + (XY)^2/2$. Eliminating the bad quartic term from $X^3 + \mathcal{N}_{2,2} + Y^5$ leaves a nonzero alternating quartic coefficient and a nonzero pure fifth coefficient. The coordinate transformations in Corollary 3.2 therefore map both five-jets to the same five-jet. Reversing the changes for the second potential gives the coordinate identification established in [Brown–Wemyss 26, Lemma 8.1]. Theorem 8.3 of that paper carries every formal tail of this five-jet to an isomorphic Jacobi algebra belonging to the fifth polynomial family in Corollary 8.14. Finally, Theorem 9.2 of [Brown–Wemyss 26], implication (2) ⇒ (3), realizes that family geometrically.

4 Growth exclusions

We set $A_f = \mathrm{Jac}(f)$ and write $J = \mathrm{rad}\, A_f$ for the Jacobson radical of A_f. Any continuous coordinate change with continuous inverse preserves the radical filtration, which ensures that the quotient dimensions $\dim_{\mathbb{C}} A_f/J^N$ are invariant under such automorphisms.

Theorem 4.1 (Zero-quartic exclusion)**.** *For every cyclic formal potential $f = x^3 + O_5$ and every $N \geq 1$,*

$$\dim_{\mathbb{C}} A_f/J^N \;\geq\; Z(N) := \sum_{k=0}^{\lfloor (N-1)/3 \rfloor} \binom{N-3k+1}{2} = \frac{N^3}{18} + O(N^2).$$

In particular $\mathrm{Jdim}\, A_f \geq 3$.

Theorem 4.2 (Both degenerate sextic axes)**.** *Let*

$$f = x^3 + y^4 + b(xy^2)^2 + c(xy)^3 + O_7, \qquad bc = 0,$$

with an arbitrary complete cyclic tail. Then $\mathrm{Jdim}\, A_f \geq 3$. *More precisely, after the weight-preserving normalization of Lemma 4.11, the following bounds hold for every positive integer N, and hence also hold for the original algebra:*

$$b = 0: \quad \dim_{\mathbb{C}} A_f/J^{3N} \geq 3\binom{N+2}{3},$$
$$c = 0: \quad \dim_{\mathbb{C}} A_f/J^{2N} \geq 2 \sum_{k=0}^{\lfloor (N-1)/2 \rfloor} \binom{N-2k+1}{2}.$$

The intersection $b = c = 0$ satisfies both bounds.

The proof of Theorem 4.1 begins with an associated graded calculation on a Zariski open set of fifth jets, followed by specialization at each finite cutoff. For the degenerate sextic axes in Theorem 4.2, splitting a pure power produces a complete quiver whose Jacobi algebra yields lower bounds for the original radical quotients.

4.1 The associated graded algebra on an open set of fifth jets

The next proposition computes the associated graded algebra for a normalized fifth jet, allowing every cyclic tail of order at least six. Subsection 4.2 identifies the open set represented by this jet and extends the dimension bound by specialization.

Proposition 4.3 (Generic fifth-jet normal form)**.** *For every cyclic $h \in O_6$, let*

$$f = \frac{x^3}{3} + xyxy^2 + \frac{y^5}{5} + h.$$

Then

$$\operatorname{gr}_J \operatorname{Jac}(f) \cong \frac{\mathbb{C}\langle x, y\rangle}{(x^2,\ y^4 + xy^2x + yxyx + xyxy)}.$$

In particular, $\operatorname{gr}_J \operatorname{Jac}(f)$ *has Hilbert series* $1/((1-t)^2(1-t^3))$.

To prove Proposition 4.3, we determine the homogeneous relations and their syzygies over $\mathbb{Q}$, and then lift these syzygies after extending scalars to $\mathbb{C}$. Let $R = \mathbb{Q}\langle x, y\rangle$ carry the standard grading by word length. We define the homogeneous elements

$$U = yxy^2 + y^2xy, \qquad R_0 = x^2, \qquad R_1 = y^4 + xy^2x + yxyx + xyxy.$$

Let $M = RpR \oplus RqR$ be the free relation bimodule with generator weights $2, 4$. The R-bimodule homomorphism $d_0 : M \to R$ sends p to R_0 and q to R_1.

Lemma 4.4 (Finite rational syzygy certificate)**.** *The kernel of d_0 is generated as an R-bimodule by*

$$\begin{aligned} s_1 &= xp - px, \\ \sigma &= x(yq - qy) + (yq - qy)x - Up + pU, \end{aligned}$$

and all Peiffer identities

$$R_i a\theta_j - \theta_i a R_j, \qquad i, j \in \{0, 1\}, \quad a \in R, \quad \theta_0 = p,\ \theta_1 = q.$$

Moreover

$$H_{R/(R_0,R_1)}(t) = \frac{1}{(1-t)^2(1-t^3)}.$$

Proof. There are two assertions to check. The critical identities generate the relation kernel, and the irreducible words determine the Hilbert series. We use the degree-lexicographic order with $x < y$. A homogeneous Gröbner basis for the two-sided ideal is

$$\begin{aligned} &x^2, \\ &y^4 + yxyx + xy^2x + xyxy, \\ &y^3xy^2x + y^3xyxy - yxyxy^3 - xy^2xy^3, \\ &y^3xyx + y^2xy^2x + y^2xyxy - yxyxy^2 - xy^2xy^2 - xyxy^3, \\ &y^2xyx + yxy^2x - xy^2xy - xyxy^2. \end{aligned}$$

The fourth basis element is redundant and is retained in the certificate. The Python verifier checks that all five elements lie in the ideal of the two original relations and resolves all seventeen overlaps and inclusions. Substituting the recorded expressions back into the free relation bimodule gives eleven nonzero critical syzygies. Their weights and multiplicities are

weight	3	6	7	8	9	10
number	1	1	1	4	3	1.

The JSON certificate expresses these eleven critical syzygies using thirty-one explicit rational multiples of s_1, σ, and Peiffer identities. The verifier checks these identities in the free relation bimodule over the polynomial free algebra, using exact rational coefficients.

The reduction from the full kernel to these finite identities proceeds by induction on the maximal leading word. In any polynomial syzygy among the relations, we compare two occurrences of relations matching at the maximal leading word. When the occurrences overlap or include one another, their difference gives a critical syzygy corresponding to an overlap or inclusion ambiguity. When the occurrences are disjoint, their difference forms a Peiffer identity. Subtracting the verified resolution of each ambiguity strictly lowers the maximal leading word,

ensuring that the reduction terminates after finitely many steps. Replacing each auxiliary Gröbner generator with its recorded polynomial expression in R_0, R_1 makes its defining relation vanish identically, while expanding its Peiffer identities produces sums of Peiffer identities for the original generators R_0, R_1. The induction and the verified identities show that s_1, σ, and the Peiffer identities generate $\ker d_0$ as an R-bimodule.

The Hilbert series. The leading monomials of the Gröbner basis yield the minimal forbidden words

$$xx, \qquad yyyy, \qquad yyxyx, \qquad yyyxyyx.$$

The deterministic prefix automaton recognizing words that avoid these four factors has ten states. The certificate provides exact polynomial numerators $n_i(t)$ with common denominator $D(t) = (1+t)(1-t)^2(1-t^3)$ satisfying the transition equations

$$n_i(t) = D(t) + t \sum_{i \to j} n_j(t).$$

At the state corresponding to the empty word, the numerator is $1 + t$. Because the linear system defined by these automaton equations admits a unique formal power series solution, these verified polynomial identities prove the asserted rational Hilbert series in every degree. □

Tensor each homogeneous component of $M \xrightarrow{d_0} R$ with $\mathbb{C}$ over $\mathbb{Q}$. In each degree this is a map between finite-dimensional rational vector spaces. Flatness of scalar extension preserves its kernel, so the displayed syzygies and their bimodule multiples generate the relation kernel over $\mathbb{C}$ as well. We write $\widehat{R} = \prod_{n \geq 0} R_n$ for the degree completion, now with ground field $\mathbb{C}$. The following lifting argument works modulo one finite power of the augmentation ideal at a time.

Lemma 4.5 (Liftable syzygies and a closed initial ideal)**.** *Let r_i be filtered formal relations with homogeneous initial relations R_i. If every homogeneous syzygy of the R_i is a sum of homogeneous syzygies admitting exact syzygy lifts for the r_i with the same initial terms, then*

$$\operatorname{gr}\big(\widehat{R}/\overline{(r_i)}\big) \cong R/(R_i).$$

Proof. Let an element of the closed relation ideal have order n, with nonzero initial form in degree n. Modulo order $n+1$, we choose a representation of this element as a sum of relation multiples. If the lowest-weight terms of this representation cancel, that cancelling component forms a homogeneous syzygy of the initial relations R_i. By hypothesis, every such homogeneous syzygy is a sum of homogeneous syzygies that admit exact lifts to syzygies of the filtered relations r_i with the same initial forms. Subtracting these exact lifts leaves the represented element unchanged while strictly increasing the minimal weight appearing in its representation. The least weight strictly increases and cannot pass n, so finitely many subtractions express the weight-n initial form as a combination of the R_i. The initial ideal of $\overline{(r_i)}$ is therefore contained in (R_i). Conversely, the initial ideal contains each R_i as the initial form of r_i. This argument applies to the topologically closed ideal because determining the initial form of an element of order n requires only a representation valid through degree n. □

Proof of Proposition 4.3. By Lemma 4.5, it suffices to lift the generators of the homogeneous relation kernel supplied by Lemma 4.4. The two cyclic derivatives of f are

$$r_x = x^2 + U + \partial_x h, \qquad r_y = R_1 + \partial_y h.$$

Assign weights $2, 4$ to the formal relation symbols p, q. The universal cyclic identity gives the exact syzygy

$$S_1 = [x, p] + [y, q], \qquad dS_1 = [x, r_x] + [y, r_y] = 0.$$

The initial term is s_1 of weight three. From this lift form a second syzygy

$$S_2 = xS_1 + S_1 x - r_x p + p r_x.$$

Indeed, direct evaluation shows $dS_2 = 0$. Cancelling its weight-four terms gives

$$S_2 = \sigma - (\partial_x h)p + p(\partial_x h).$$

Since $\partial_x h$ has order at least five, the last two terms have weight at least seven. Thus S_2 lifts the weight-six syzygy σ. Replacing R_i by r_i lifts every Peiffer identity. The scalar-extension argument above Lemma 4.5 shows that these lifts cover every homogeneous syzygy over $\mathbb{C}$. The lifting lemma therefore identifies the initial ideal with (R_0, R_1) and proves the asserted graded isomorphism.

□

4.2 Specialization to all fifth jets

Proof of Theorem 4.1. We first identify the Zariski open set of fifth jets to which Proposition 4.3 applies. The relation matrices at a finite cutoff will then extend the resulting Hilbert-function bound to every fifth jet. Scaling the potential allows us to assume that its cubic part is $x^3/3$. In degree five, an order-three coordinate correction of x removes every necklace containing xx. The remaining fifth jet has the form

$$\frac{x^3}{3} + Axyxy^2 + Bxy^4 + Cy^5.$$

When $C \neq 0$, we apply the linear shear $y \mapsto y + tx$ and remove the cyclic xx terms generated by this substitution. The surviving coefficients become

$$A' = A + 2Bt + 5Ct^2, \qquad B' = B + 5Ct, \qquad C' = C.$$

The shear leaves the discriminant $B^2 - 5AC$ invariant. On the open set defined by

$$C \neq 0, \qquad B^2 - 5AC \neq 0,$$

we take $t = -B/(5C)$. This choice kills the xy^4 term, since $B' = 0$, while leaving $A' = -(B^2 - 5AC)/(5C) \neq 0$. Rescaling x, y and the overall potential recovers the exact fifth jet of Proposition 4.3. No quartic terms are introduced, and the remaining O_6 tail is arbitrary. The coordinate changes induce continuous isomorphisms of the completed Jacobi algebras. Multiplying the potential by a nonzero scalar leaves its derivative ideal unchanged. Both operations preserve the radical quotient dimensions, so Proposition 4.3 gives the same truncated dimensions throughout this open set.

Specialization at a finite cutoff. Fix $N \geq 1$. We allow the coefficients to vary through degree $\max(N, 5)$, so that the conditions on the fifth jet remain defined even for small N. Modulo $\mathfrak{m}^N$, the ambient algebra is finite-dimensional, and the two-sided multiples of the cyclic derivatives define a relation matrix whose entries are polynomials in the potential coefficients. On the dense open set of fifth jets, Proposition 4.3 proves that the quotient dimension $\dim_{\mathbb{C}} A_f/J^N$ is constant, which means that the relation matrix has constant rank r. Every minor of size $r+1$ vanishes on this open set and hence vanishes identically throughout the parameter space. At any specialized potential, the relation matrix can therefore only have rank at most r. Because the quotient dimension is the ambient dimension minus the rank of the relation matrix, every specialization has quotient dimension at least the generic value. The cyclic derivatives of potential terms beyond degree N lie in $\mathfrak{m}^N$, so this establishes the lower bound for the complete formal potential.

Finally, the degree-n coefficients of the Hilbert series are given by

$$[t^n]\frac{1}{(1-t)^2(1-t^3)} = \#\{(a,b,c) \in \mathbb{Z}^3_{\geq 0} : a + b + 3c = n\}.$$

Summing over $n < N$ gives $Z(N)$. The leading quadratic terms of the binomial coefficients sum to $N^3/18 + O(N^2)$, which completes the proof of Theorem 4.1. □

4.3 Two complete-quiver growth bounds

Let Q_q denote the quiver with q vertices and one arrow for each ordered pair of vertices, including a loop at each vertex. We write $\mathfrak{a}$ for its arrow ideal. The two degenerate sextic axes in Theorem 4.2 correspond to the pairs $(q, m) = (3, 3)$ and $(2, 4)$, where the potential has order at least m in the arrow ideal.

Proposition 4.6 (Uniform quiver bounds)**.** *Over any characteristic-zero extension of $\mathbb{C}$, the following hold. For every potential $W \in \mathfrak{a}^3$ on Q_3,*

$$\dim \mathrm{Jac}(W)/\mathfrak{a}^N \geq 3\binom{N+2}{3}.$$

For every potential $W \in \mathfrak{a}^4$ on Q_2,

$$\dim \mathrm{Jac}(W)/\mathfrak{a}^N \geq 2\sum_{k=0}^{\lfloor (N-1)/2 \rfloor} \binom{N-2k+1}{2}.$$

The potentials may be formal and need not be homogeneous.

The estimates follow by varying the potential coefficients about two explicit homogeneous potentials. At each finite cutoff, vanishing in cohomological degree minus one on the special fiber gives flatness over the parameter ring.

We employ two gradings on the Ginzburg dg algebra. The arrows, opposite arrows, and vertex loops are assigned cohomological degrees $0, -1, -2$, respectively, and the zeroth cohomology computes the completed Jacobi algebra. In addition, positive internal weights yield finite truncations. At the two homogeneous potentials constructed below, the differential preserves internal weight exactly.

Lemma 4.7 (A finite flat-complex criterion)**.** *Let $(T, \mathfrak{n}, k)$ be a Noetherian local ring and let P be a nonpositive complex of flat T-modules. Suppose $H^0(P)$ is finite and $H^{-1}(P \otimes_T k) = 0$. Then $H^0(P)$ is finite free over T.*

Proof. It suffices to show that $\operatorname{Tor}_1^T(H^0(P), k) = 0$. Put $K = \operatorname{im}(P^{-1} \to P^0)$ and $B = H^0(P) = P^0/K$. Given an element of $K \otimes_T k$ that vanishes in $P^0 \otimes_T k$, lift it to $P^{-1} \otimes_T k$. The image of this lift in $P^0 \otimes_T k$ is zero, so the lift is a cycle. By the hypothesis that $H^{-1}(P \otimes_T k) = 0$, this cycle is a boundary coming from $P^{-2} \otimes_T k$, which implies that its image in $K \otimes_T k$ vanishes. Hence the canonical map $K \otimes_T k \to P^0 \otimes_T k$ is injective. From the short exact sequence $0 \to K \to P^0 \to B \to 0$, the flatness of P^0 now gives $\operatorname{Tor}_1^T(B, k) = 0$.

Take a minimal surjection from a finite free T-module onto B. Its kernel is finitely generated by Noetherianity and has zero reduction modulo $\mathfrak{n}$ by the Tor vanishing and minimality. Nakayama's lemma then implies that the kernel is zero, which proves the claim. □

Lemma 4.8 (Exact homogeneous seeds)**.** *There are homogeneous potentials W_3 of degree three on Q_3 and W_2 of degree four on Q_2 whose Ginzburg differential graded algebras have zero negative cohomology. Their Jacobi algebras have total Hilbert series*

$$H_3(t) = \frac{3}{(1-t)^3}, \qquad H_2(t) = \frac{2}{(1-t)^2(1-t^2)}.$$

For a seed of degree m, assign internal weight one to every arrow; opposite generators have weight $m-1$ and the degree-minus-two vertex generators have weight m.

Proof. We perform the initial calculations within the graded path algebras, after which completion takes the direct product of the homogeneous components. Because the differential of each homogeneous seed preserves the internal weight grading, these calculations simultaneously determine the completed algebras in each internal weight and at every finite internal-weight cutoff.

(i) The cubic potential on Q_3. For $q = 3$, let the cyclic group of order three act on the polynomial ring $\mathbb{C}[X, Y, Z]$ with characters $0, 1, 2$. The corresponding McKay quiver is the complete 3-vertex quiver Q_3. The McKay correspondence lifts the invariant cyclic potential $XYZ - XZY$ to a quiver potential whose Jacobi algebra is the skew group algebra $\mathbb{C}[X, Y, Z] \rtimes C_3$. Explicitly, taking vertex indices modulo three, let $a_i : i \to i$ denote the loops, $b_i : i \to i+1$ the forward arrows, and $c_i : i \to i-1$ the backward arrows. The cubic potential is

$$W_3 = \sum_i (a_i b_i c_{i+1} - a_i c_i b_{i-1}).$$

The cyclic derivatives give the three commutation relations with their vertex labels. The Ginzburg differential graded algebra is the cobar algebra of the exterior Koszul coalgebra on three generators, crossed with C_3. The Koszul resolution identifies its augmentation to the skew polynomial algebra as a quasi-isomorphism. Because the volume element is invariant, the degree-minus-two generators are vertex loops. Counting PBW monomials gives the first Hilbert series.

(ii) The quartic potential on Q_2. For $q = 2$, consider the order-two group C_2 acting by the involution $X \mapsto X$ and $Y \mapsto -Y$. The McKay lift of the invariant potential $\frac{1}{2}[X, Y]^2$ is a quartic potential on the two-vertex quiver Q_2. The resulting Jacobi algebra is

$$U(\mathfrak{h}_3) \rtimes C_2, \qquad [X, Y] = Z, \quad [X, Z] = [Y, Z] = 0,$$

where $\deg X = \deg Y = 1$ and $\deg Z = 2$. The involution sends (X, Y, Z) to $(X, -Y, -Z)$. The second Hilbert series follows from PBW. To prove the required vanishing of negative cohomology, we realize this algebra through a three-generator potential and then eliminate a contractible pair. Regard Z as a third free generator and use the stabilized potential

$$\widetilde{W} = Z[X, Y] - \tfrac{1}{2}Z^2.$$

The associated Ginzburg differential graded algebra is the cobar construction of the Chevalley–Eilenberg coalgebra of $\mathfrak{h}_3$, crossed with C_2. The cobar augmentation to $U(\mathfrak{h}_3)$ is an equivariant quasi-isomorphism. Indeed the PBW filtration reduces it to the exterior Koszul calculation for the abelian three-dimensional Lie algebra. Since the differential preserves the positive internal weights, the conclusion holds separately in each weight.

The remaining step is to return to the quartic Ginzburg algebra. Denote the opposite generators of the stabilized algebra by p, q, r, so that

$$dp = [Y, Z], \quad dq = [Z, X], \quad dr = [X, Y] - Z, \quad d\zeta = [X, p] + [Y, q] + [Z, r].$$

Set

$$Z' = Z - [X, Y], \quad p' = p + [Y, r], \quad q' = q + [r, X], \quad \zeta' = \zeta + r^2.$$

Then $dr = -Z'$, $dZ' = 0$, and

$$dp' = [Y, [X, Y]], \quad dq' = [[X, Y], X], \quad d\zeta' = [X, p'] + [Y, q'].$$

Thus the stabilized differential graded algebra is the quartic Ginzburg algebra with the contractible pair (r, Z') freely adjoined. Projection onto the quartic differential graded algebra is a quasi-isomorphism. In each internal weight, the words involving the pair form tensor complexes with a contractible factor. The coordinate change respects both the weights and the involution, proving the required vanishing. □

Proof of Proposition 4.6. For a fixed arrow cutoff we first obtain a free family near the chosen homogeneous potential. The polynomial relation matrix then gives the dimension bound everywhere. Fix N and $(q, m) \in \{(3, 3), (2, 4)\}$. Only potential terms in degrees $m, \dots, N$ affect $\mathrm{Jac}(W)/\mathfrak{a}^N$. Vary their cyclic coefficients independently about W_q over $T = \mathbb{C}[[e_1, \dots, e_M]]$. Give the arrows, opposite arrows and vertex loops of the relative Ginzburg differential graded algebra over T internal weights $1, m-1, m$, respectively. The differential never lowers internal weight. We can therefore discard weights at least N in every cohomological degree, leaving a complex of finite free T-modules.

At the homogeneous potential W_q, the differential preserves internal weight exactly. Its truncated complex is the direct sum of the full weight complexes below N, so Lemma 4.8 gives vanishing in cohomological degree minus one. In cohomological degree zero, the cohomology of the relative complex is the truncated Jacobi algebra $\mathrm{Jac}(W_T)/\mathfrak{a}^N$. The complex has finite free terms and its zeroth cohomology is finite. Lemma 4.7 therefore makes this cohomology free over T, with rank given by its special fiber.

The differential into cohomological degree zero is a matrix of polynomials in the potential coefficients. If its generic rank is r, every $(r+1) \times (r+1)$ minor vanishes as a formal series at the seed and hence vanishes identically. At every coefficient point the rank is at most r, so the quotient dimension is at least the dimension at the seed. Translation by the seed covers the full coefficient space. Summing the coefficients of the two Hilbert series in Lemma 4.8 gives the asserted bounds. The minor identities survive scalar extension, and terms of degree greater than N do not affect this quotient. □

4.4 Growth transfer by partial morsification

We now deform the pure power so that its roots become distinct. Each root provides a vertex, and the matrix blocks of the remaining variable generate the arrows between these vertices. To establish a lower bound for the original Jacobi algebra, we compare the special fiber of the deformed family with a generic fiber that retains all q roots as distinct vertices.

Lemma 4.9 (Pure-power splitting and nilpotent quotients)**.** *Let*

$$f = P(u) + \frac{v^{q+1}}{q+1} + g(u, v),$$

where $P \in u^m\mathbb{C}[[u]]$ and every word of g contains at least m occurrences of u, with $m \geq 3$. Suppose that, for each fixed number of u's, only finitely many words of g occur; the same conclusion holds if the substitutions below converge in the corresponding adic topology. Put $A = \mathrm{Jac}(f)$, $I = (u)$ and $J = \operatorname{rad} A$. For each N there is a finite $\mathbb{C}[[s]]$-module M_N whose special fiber is A/I^N and whose generic fiber is

$$\mathrm{Jac}(W)/\mathfrak{a}^N$$

for a potential $W \in \mathfrak{a}^m$ on Q_q over $K = \mathbb{C}((s))$. In particular,

$$\dim_{\mathbb{C}} A/J^{qN} \;\geq\; \dim_{\mathbb{C}} A/I^N \;\geq\; \dim_K \mathrm{Jac}(W)/\mathfrak{a}^N.$$

Proof. We construct a finite module over the parameter discrete valuation ring before passing to the generic fiber. We then identify that generic fiber using all q roots of the split pure power.
(i) The finite module over D. Take $F_s = f - s^q v$ over $D = \mathbb{C}[[s]]$ and let M_N be its relative Jacobi quotient modulo I^N, completed in the total (s, u, v)-adic topology. Its special fiber is A/I^N. Since $A/I = \mathbb{C}[[v]]/(v^q)$, we have $J^{qN} \subset I^N$. Put $K = \mathbb{C}((s))$ and $\lambda_i = s\zeta_q^i$ for a primitive root ζ_q.

To keep the full generic support, we first work over the parameter ring D. The relative relation $v^q - s^q = -\partial_v g \in I^m$ gives

$$(v^q - s^q)^{\lceil N/m \rceil} = 0 \qquad (\mathrm{mod}\ I^N).$$

Under the finite-word hypothesis the truncated derivative relations are polynomials, since only terms with at most N occurrences of u contribute. We form the polynomial quotient modulo I^N over $\mathbb{C}[[s]]$. Its reduction modulo I is the finite free algebra $\mathbb{C}[[s]][v]/(v^q - s^q)$. Each subquotient I^j/I^{j+1} is spanned by products of j copies of u separated by representatives of a basis of this algebra. The polynomial quotient is therefore finite

over the discrete valuation ring $\mathbb{C}[[s]]$. Modulo s its augmentation ideal is nilpotent, which implies that the total-adic and s-adic topologies are cofinal. The quotient is already complete and equals M_N.

For convergent substitutions we take the remainder of each string of v's between successive u's modulo the displayed monic polynomial in v. The lower coefficients lie in (s), so the remainders converge s-adically and establish the same finite-module property.

(ii) The completed algebra at all q roots. Finiteness of M_N over the parameter discrete valuation ring D bounds the dimension of the generic fiber by that of the special fiber. To identify the generic fiber, we invert s. Modulo I we obtain K^q. The powers of the factors $v - \lambda_i$ are pairwise coprime for the q distinct roots, so their Chinese remainder idempotents lift all q vertices as polynomials in v. These idempotents commute with v, although they need not commute with u.

We use these spectral idempotents before imposing the Jacobi relations. We complete the free algebra over K at the kernel of $K\langle u, v\rangle \to K^q$. Successive lifting in the $(v^q - s^q)$-adic topology gives the orthogonal idempotents inside the closed commutative subalgebra generated by v. Relative to this copy of $K[v]/(v^q - s^q) \cong \prod_{i=1}^{q} K$, the completed free algebra has generators

$$u_{ij} = e_i u e_j \quad (1 \le i, j \le q), \qquad t_i = e_i(v - \lambda_i)e_i \quad (1 \le i \le q).$$

These are the generators of the completed path algebra on Q_q with q vertices, the q^2 arrows u_{ij}, and the additional q vertex loops t_i. The inverse substitution sends u to $\sum_{i,j} u_{ij}$ and v to $\sum_i(\lambda_i e_i + t_i)$. The spectral projectors of this last element recover the vertex idempotents.

(iii) Elimination of the quadratic loops. The expansion of the pure v-potential $\frac{v^{q+1}}{q+1} - s^q v$ at each root λ_i has nondegenerate quadratic part in the loop t_i, because

$$(v^q - s^q)'|_{v=\lambda_i} = q\lambda_i^{q-1} \neq 0.$$

Every other term in f contains at least m copies of u. We complete the squares successively in the loops t_i while fixing the arrows u_{ij}, following the noncommutative splitting lemma. The critical loop values satisfy $t_i \in (u_{ij})^m$. Substituting them therefore produces a quiver potential W of arrow order at least m, that is, $W \in \mathfrak{a}^m$.

Before eliminating the quadratic loops, differentiation with respect to the arrows and loops gives the matrix blocks of $\partial_u F_s$ and the diagonal blocks of $\partial_v F_s$. The splitting imposes the loop equations and leaves the derivatives of W in the surviving arrows. Modulo these relations, the universal cyclic identity gives $[v, \partial_v F_s] = 0$. On any off-diagonal block with $i \neq j$, the commutator operator

$$z \longmapsto (\lambda_i + t_i)z - z(\lambda_j + t_j)$$

is invertible. Its constant term is the scalar $\lambda_i - \lambda_j = s(\zeta_q^i - \zeta_q^j) \neq 0$, and the remaining operator is topologically nilpotent. Thus the off-diagonal blocks of the derivative vanish in the quotient too. The resulting isomorphism preserves the ideal generated by the arrows u_{ij}.

(iv) The dimension comparison. Quotienting by the N-th power of the arrow ideal yields a finite-dimensional K-algebra. The substitutions identify the generic fiber of M_N with the arrow-ideal quotient $\mathrm{Jac}(W)/\mathfrak{a}^N$. Indeed the ideal generated by u and $v^q - s^q$ is nilpotent modulo I, and $I^N = 0$, so completion at the full semisimple quotient preserves this finite fiber. The finiteness of M_N over the discrete valuation ring D establishes the dimension inequality $\dim_{\mathbb{C}} A/I^N \ge \dim_K \mathrm{Jac}(W)/\mathfrak{a}^N$. Combining this inequality with the inclusion $J^{qN} \subset I^N$ completes the proof of both bounds. □

Remark 4.10. The finite-word hypothesis holds for the normalized potentials below. In a cyclic word avoiding xx and yyy, the number of y's is at most twice the number of x's, and the number of x's is at most the number of y's. A bound on either letter count therefore bounds the word length.

4.5 Application to the sextic axes

Lemma 4.11 (Weighted normalization)**.** *Give x, y weights $4, 3$. Every $x^3 + y^4 + O_5$ is cyclically right-equivalent to a potential whose higher necklaces avoid cyclic xx and cyclic yyy. Below weight 24 its only possible higher necklaces are*

$$a\,xyxy^2 \quad (\mathrm{wt}\,17), \qquad b(xy^2)^2 \quad (\mathrm{wt}\,20), \qquad c(xy)^3 \quad (\mathrm{wt}\,21).$$

For $x^3 + y^4 + b(xy^2)^2 + c(xy)^3 + O_7$, the normalization leaves b, c unchanged and creates no weight-17 term.

Proof. Lemma 2.14 gives the convergent weighted normalization. For the displayed sextic jet there are no earlier bad terms to alter its coefficients. The terms of ordinary order O_7 in weights $21, 22, 23$ are all bad necklaces. Removing them changes no good necklace of those weights, so the normalized remainder still has ordinary order at least seven. □

Proof of Theorem 4.2. Lemma 4.11 preserves the two sextic coefficients and removes the bad necklaces. For these normalized tails, a bound on either letter count bounds the total word length. This verifies the finite-word hypothesis of Lemma 4.9. It remains to count occurrences of the retained variable on each axis.

First suppose $b = 0$. Every mixed word in the normalized tail contains at least three occurrences of x. Indeed, with only two x's a good necklace has at most four y's, hence length at most six. The only such sextic is $(xy^2)^2$, absent because $b = 0$, and the weight-seventeen necklace $xyxy^2$ is absent as well. After rescaling the coordinates we have

$$f = \frac{x^3}{3} + \frac{y^4}{4} + g, \qquad \text{where every word of } g \text{ contains at least three } x\text{'s.}$$

We apply Lemma 4.9 with retained variable $u = x$, split variable $v = y$, and parameters $(q, m) = (3, 3)$. The uniform quiver bound of Proposition 4.6 yields

$$\dim_{\mathbb{C}} A/J^{3N} \geq 3\binom{N+2}{3}.$$

If $c = 0$, every mixed term of the normalized tail contains at least four occurrences of y. A good necklace with at most three y's has at most three x's and thus length at most six. The only possible sextic is $(xy)^3$, absent because $c = 0$. We now retain $u = y$ and split $v = x$. Applying Lemma 4.9 with parameters $(q, m) = (2, 4)$ together with Proposition 4.6 yields

$$\dim_{\mathbb{C}} A/J^{2N} \geq 2 \sum_{k=0}^{\lfloor (N-1)/2 \rfloor} \binom{N-2k+1}{2}.$$

Both bounds grow cubically in N, so $\operatorname{Jdim} A_f \geq 3$ on either axis. At their intersection $b = c = 0$ both bounds apply. □

5 Proof of the realization theorem

Proof of Theorem 1.3. By Proposition 2.1 it suffices to realize the potentials $x^3 + O_4$ satisfying the numerical growth bound. We apply the quartic classification of Proposition 2.3.

If $\Delta \neq 0$, Theorem 2.12 constructs a length-three arc for each potential in this stratum and separates the small contractions from the divisorial contractions.

For an alternating quartic, Theorem 3.1 demonstrates that any potential with $\operatorname{Jdim} A_f \leq 1$ must have a nonzero pure fifth coefficient. We normalize the five-jet using Corollary 3.2. The coordinate change $X = y + x^2$, $Y = x$ preserves ordinary filtration order in both directions, so in the new coordinates the potential takes the form $F_p + h$ with $p(u) = u^4$ and $h \in O_6$. Theorem 3.3 proves finite-dimensionality, and Proposition 3.8 realizes the algebra geometrically as a length-five flop.

For the pure y^4 quartic orbit, Lemma 4.11 isolates the surviving higher coefficients in weights $17, 20, 21$. If the weight-17 coefficient is nonzero, Theorem 2.15 realizes its Jacobi algebra by a length-four contraction. If the weight-17 coefficient vanishes, Theorem 4.2 excludes both degenerate sextic axes $b = 0$ and $c = 0$, leaving exactly the open pair of nonzero coefficients realized in length six by Theorem 2.16. The remaining quartic orbit consisting of potentials with zero quartic jet is excluded by the cubic growth bound of Theorem 4.1.

Generic surface types. Varying the hyperplane section in the fixed threefold deforms its distinguished partial-resolution surface while preserving the scheme-theoretic fiber. The generic Du Val diagram is a Dynkin subgraph of the distinguished diagram with the same retained-cycle multiplicity. In the length-three E_6 family no proper Dynkin subgraph supports multiplicity three. Thus the generic singularity type is cE_6 for both the small contractions and the divisorial contractions. For the small primitive contractions of lengths four, five, and six, the invariant-theoretic criteria of Katz and Morrison [Katz–Morrison 92, Section 8] establish the generic singularity types as cE_7, cE_8, cE_8, respectively.

Under $\operatorname{Jdim} \leq 1$ the surviving Type E branches are therefore

normalized branch	construction	generic surface type
$\Delta \neq 0$	marked length 3	cE_6
$(xy)^2$ quartic, nonzero pure fifth	marked length 5	cE_8
y^4 quartic, $a_{17} \neq 0$	marked length 4	cE_7
y^4 quartic, $a_{17} = 0$, $b_{20}c_{21} \neq 0$	marked length 6	cE_8

The growth bounds exclude every other branch.

In lengths three, four, and six, Lemma 2.6 identifies the completed Jacobi algebra with the normalized basic contraction algebra of the geometric model. For length five, Lemma 3.9 identifies the closed Jacobi ideal with

that of the polynomial truncation utilized in Proposition 3.8. All continuous coordinate changes and scalar rescalings preserve the completed Jacobi algebra up to isomorphism.

Together with the Type A and Type D cases established in Proposition 2.1, these constructions satisfy Definition 1.1 for every algebra in the theorem. The finite-dimensional algebras are realized by flopping contractions and the growth-one algebras by divisor-to-curve contractions. □

6 Computational certificates

The computations use exact arithmetic. Starting in the manuscript directory, run

```
cd tools/numerical_realisation_certificates
python3 verify_cyclic_reductions.py
python3 verify_zero_quartic.py
python3 nr_B5_generic_certificate.py --output ../../build/nr_B5_rerun.json
```

The first two programs use the Python standard library. The third uses the versions of SymPy and mpmath listed in `requirements-certificates.txt` in the manuscript directory.

For Lemma 2.13, the first script reduces each of the ten overlap ambiguities in both reduction orders. For Theorem 2.16 it forms the ordinary and cyclic relation matrices of B_6, computes their exact ranks over the rationals, and checks the seven proposed cocenter generators.

The second script reads the adjacent `zero_quartic_certificate.json`. It checks the five original-ideal identities, the seventeen overlap and inclusion ambiguities, and the thirty-one summands representing the eleven critical syzygies in Lemma 4.4. It also verifies the polynomial identities giving the rational Hilbert series of the prefix automaton. Lemma 4.5 passes from the homogeneous relations to formal relations.

The third script recomputes both finite specializations in Lemma 3.4. The supplied `nr_B5_generic_certificate.json` records the monomial basis words, the vector space dimensions 40 and 25, and the regular trace determinant and center dimension at $S = 1$. These verified values supply the finite inputs to the filtered argument of Theorem 3.3.

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Neither the existence nor the priority date of this manuscript should discourage a wholly independent human proof of these results. If a subsequent work obtains them through an argument developed wholly independently by humans, I claim no special priority or credit for the proof ideas beyond standard bibliographic acknowledgement of prior or concurrent results.

If later work makes substantive use of the results, constructions, or proof strategies in this manuscript, ordinary scholarly citation is appropriate regardless of the tools used. In particular, I would appreciate explicit attribution when closely related arguments appear in AI-assisted work, since the origin of ideas generated through related language models can otherwise be difficult to trace.

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