Numerical realisation of marked Gopakumar–Vafa spectra

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Abstract

Let $X \to (Y, 0)$ be a flopping contraction of a smooth complex threefold to an isolated cDV point, with its exceptional curve classes marked. We determine which finitely supported functions on the curve lattice occur as its complete genus-zero Gopakumar–Vafa spectrum as the contraction varies. The resulting criterion is entirely intrinsic to the marked support and its integer values, testing contact orders and linear spans of root forms; every accepted input produces an equivariant polynomial Cartan arc and hence a local algebraic realisation.

In arbitrary curve rank, the classical criterion covers all classical supports. From rank five onwards the exceptional list consists of the full E_6, E_7, E_8 root supports and ten attachment families. Another 27 signed configurations account for the remaining 211 exceptional marked support types in ranks two to four. As a label-free consequence, outside the full reduced root supports, full BC_r, and the spin extension of B_3, every multicurve spectrum has at least $2r - 2$ entries equal to one, and this bound is sharp in every curve rank $r \geq 2$.

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1 Introduction

A generic perturbation of a smooth threefold flopping contraction separates its exceptional fibre into isolated rational curves with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and the genus-zero Gopakumar–Vafa (=GV) invariant n_β counts those of class β. On the deformation base of a partial resolution, let $P_\beta = 0$ be the reduced discriminant component belonging to β. A classifying arc γ through the origin then satisfies $n_\beta = \operatorname{ord}_t P_\beta(\gamma(t))$ [5, Theorem 2.6 and (3.A)]. We address the converse question: given the curve classes and a positive integer on each, when does a single arc attain all of these intersection orders while yielding a smooth threefold?

1.1 The marked realisation problem

Fix the lattice $\Lambda = \bigoplus_{i=1}^{r} \mathbb{Z}c_i$ with its ordered basis and positive monoid $\Lambda_{\geq 0} = \sum_i \mathbb{Z}_{\geq 0} c_i$. A marked input is a finitely supported function $n : \Lambda_{\geq 0} \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ with $B = \{\beta : n_\beta > 0\}$. Its realisation is a projective small crepant contraction of complex analytic or algebraic germs

$$X \longrightarrow (Y, 0)$$

with X a smooth threefold, 0 an isolated cDV point, and ordered irreducible exceptional curves of classes $c_1, \ldots, c_r$. Its complete genus-zero GV spectrum must equal n: labels β and 2β carry separate entries and all classes outside B have value zero.

Theorem 2.1 gives a numerical test for this existence problem. When the multiplicities pass the test, we construct one polynomial arc whose intersection orders give all of them.

For one curve the list is given in Section 2.1.1. With two or more curves, an order prescribed on one divisor can force the order on another. For example, on the full A_2 resolution the three classes are $c_1, c_2, c_1 + c_2$ and their equations are $x, y, x + y$. The arc $(x(t), y(t)) = (t, -t + t^3)$ gives the three integers $(1, 1, 3)$: the linear terms cancel in the sum. The triple $(1, 2, 3)$ is impossible. If x starts in degree one and y starts in degree two, $x + y$ still has a linear term. In general the minimum of the three orders must occur at least twice, and every such triple occurs [6, Corollary 1.8].

1.2 Relation to earlier work

For one contractible rational curve C, Bryan, Katz and Leung expressed its local Gromov–Witten contributions in terms of the Hilbert-scheme multiplicities of thickenings of C [1, Theorem 1.5]. Katz identified these integers with the local genus-zero invariants defined using stable sheaves [2, Proposition 3.3]. Collinucci, De Marco, Sangiovanni and Valandro computed the invariants in explicit families of simple flops of lengths one through six [3, §4]. The full list we use in rank one comes from the classification of smooth simple flops by Brown and Wemyss [10, Theorems 1.3–1.4 and Corollary 9.11]. Their earlier examples of analytically distinct flops with the same GV invariants show why a numerical spectrum can have several realisations [9].

For a generic contractible A_n chain, Bryan, Katz and Leung obtain one $(-1, -1)$-curve in each interval class after deformation [1, §2.2]. For a Du Val section of arbitrary ADE type, Katz and Morrison's simultaneous partial resolution gives the deformation family and identifies the induced deformations at its Du Val points [4, §6, Theorem 3]. Nabijou and Wemyss identify the components of the discriminant by restricted positive roots. For a given flop they express n_β as the number of intersections of a perturbed classifying map with the component labelled by β [5, Theorem 2.6, (3.A) and Corollary 3.1].

Zhang studies simultaneous numerical constraints for cA_n resolutions. The value on an interval of the exceptional chain is at least the minimum of the values on its simple classes, with equality forced when that minimum is unique [6, Theorem 1.7]. His complete cA_2 classification gives exactly the repeated-minimum condition for the three divisors above when the invariants are finite [6, Theorem 1.3 and Corollary 1.8].

The same numbers have an algebraic interpretation. For a fixed smooth flop, Bi constructs a finite free marked deformation of its basic contraction algebra with split semisimple generic fibre. Each class $\beta = (\beta_i)$ contributes n_β matrix blocks of size $\sum_i \beta_i$, and the projector for C_i has rank β_i on each block. Every finite free formal deformation with geometrically semisimple generic fibre gives the same marked type [7, Theorem 1.2(i)–(ii)]. For the three labels $(1, 0), (0, 1), (1, 1)$ with multiplicities a, b, c, the algebra therefore has dimension $a+b+4c$, and its cocenter has dimension $a+b+c$ [7, §4.3.7]. Here a, b, c are the proposed input. To decide which triples occur we must go back to the three divisor equations and impose smoothness on their common arc.

Goresky, Kottwitz and MacPherson give nonemptiness criteria for prescribed valuations of individual roots, both on the Cartan and on its Weyl-twisted forms [8, Propositions 3.4.1 and 4.8.2]. In the present problem the input gives one normalised sum of orders for each restricted-root fibre, and smoothness prescribes the internal root orders. Theorem 2.1 determines which collections of these sums a single arc can attain. Its conditions on the marked integers specify when two fibres can share a leading cancellation, and an accepted collection yields a polynomial arc realising all the sums simultaneously.

1.3 The role of the complete support

Two rank-two supports illustrate the force of the word *complete*. Let w, q be a lattice basis. Take the three labels $w, 2w, 3w$ and the five labels $q + jw$, $-2 \le j \le 2$, with signs chosen in the

marked chamber. The mixed values are fixed. On this eight-label support row 2c of Table A.1 requires $\min(n_{2w}, n_q) = 3$. In particular $n_{2w} = 4$ rules out $n_q = 5$. Now include $2q$ as a ninth label. Theorem 6.13 realises $(n_w, n_{2w}, n_{3w}) = (7, 4, 1)$ together with $(n_q, n_{2q}) = (5, 1)$. Section 6.3.5 constructs the two radial spectra by prescribing independent terms of the Cartan arc.

The classification also gives obstructions that survive forgetting the labels. In curve rank $r \geq 2$ two multiplicities must coincide (Theorem 5.8). Outside the full reduced root supports, full BC_r, and the spin extension of B_3, at least $2r - 2$ multiplicities equal one, and this bound is attained (Theorem 5.1).

For instance, every complete spectrum with seven nonzero values has at least three entries equal to one. The list $(6, 5, 3, 2, 2, 2, 1)$ has only one. Section 5.1.4 gives the argument for all assignments of its entries to curve classes.

1.4 Organisation of the paper

To prove Theorem 2.1, we fix a Du Val section and track each root fibre along the lifted Cartan arc. Smoothness prescribes the first nonzero order on each internal root. By the last of these mandatory degrees, some roots over β have already contributed a fixed amount b_β to the sum n_β; any surviving roots over β form a single Coxeter orbit and therefore share a common order (Lemma 2.3). The residual difference $n_\beta - b_\beta$ determines how long that orbit must continue to vanish. At each subsequent degree a single coefficient vector must annihilate every root exiting at a higher degree and be nonzero on every root exiting now; compatible choices at the finitely many attained degrees sum to a polynomial arc.

Three structural phenomena emerge from this analysis: on full BC_r, realisability factors canonically into independent contact classes and is governed by a sharp three-block integer test (Theorems 3.1 and 3.3), whereas general supports admit global obstructions invisible on every proper root flat (Section 5.5); outside the full supports and the spin extension of B_3, every multicurve spectrum has at least $2r - 2$ unit entries and a repeated multiplicity (Theorems 5.1 and 5.8); and on the doubled-short G_2 support, two distinct ambient ADE presentations produce disjoint numerical phases (Section 6.3.6), showing that the smallest-rank ambient presentation need not dominate.

The paper is organised into six sections and three appendices. Section 2 states Theorem 2.1, derives the Cartan–GV fibre formula alongside a worked two-curve example, and derives the integer image of a fixed presentation in Section 2.3.3. Section 3 resolves all classical supports in arbitrary rank. Section 4 classifies all supports from rank five onwards, illustrates the low-rank contact plane on row 2e, and proves Theorem 2.1 by the two inclusions in (4.2). Section 4.5 exhibits a spectrum which distinguishes two ambient types in row 3f. Section 5 derives label-free obstructions, delimits quotients and proper flat restrictions, and determines the effective affine monoid cover and box counts on a fixed support. Section 6 collects the explicit root-lattice constructions and exhaustion proofs for the individual exceptional attachment families. Finally, Appendix A tabulates the 27 low-rank exceptional configurations, Appendix B collects the explicit polynomial witnesses and finite verification certificates, and Appendix C provides a glossary of notation, tests, and support families. The classical results, the classification for $r \geq 5$, and the classical part of the $2r - 2$ unit bound have conceptual proofs. The additional low-rank classification uses the finite support enumeration in Appendix B.2 and the unbounded integer comparison in Proposition 4.3. Appendix B.3.4 gives both calculations and the independent checks of the explicit models.

1.5 Notation and conventions

We work with complex threefold germs. For any full-rank finite positive configuration $B \subset \Lambda$, put

$$M_B = \sum_{\beta \in B} \beta\beta^{\mathsf{T}}, \qquad (x, y)_B = x^{\mathsf{T}} M_B^{-1} y.$$

This inverse moment metric is independent of the choice of lattice coordinates, and we use its squared-length partitions to distinguish support frames intrinsically. Every support identification is checked against the entire vector set in its lattice. A regular linear functional selects a positive half B of a symmetric signed configuration. In any signed formula, $n_{-\beta} = n_\beta$ denotes the value at the positive representative; the input function itself is defined on the positive monoid $\Lambda_{\geq 0} \setminus \{0\}$.

2 The Cartan–GV interface and core lemmas

In this section we state the main realisation theorem, derive the Cartan–GV fibre formula linking root orders to Gopakumar–Vafa invariants, illustrate the mechanism on a concrete two-curve partial resolution, and establish the general coefficient-avoidance, surviving-orbit, and finite-recognition tools used throughout the paper.

2.1 The realisation theorem

Let $c_1, \dots, c_r$ be the ordered curve classes of Section 1.1, let B be the support of n, and put $B^{\pm} = B \cup (-B)$. We first determine which supports and which multiplicities can occur together.

Theorem 2.1 (Complete marked numerical realisation)**.** *Let $(\Lambda, (c_i), B, n)$ be a complete marked input as in Section 1.1, with $r \geq 2$. A projective small crepant contraction of a smooth complex threefold to an isolated cDV germ realises this spectrum if and only if all c_i belong to B and at least one of the following tests succeeds under an integral identification preserving the entire signed support and the induced ordered chamber marking:*

1. *the classical test of Section 3 or one of the criteria for the support families in Section 4.2, including the nine-label family of Section 6.3.5;*
2. *a row of Appendix A, with every indicated constant, integer condition and contact test satisfied.*

All coefficients on omitted positive labels are zero. Each accepted tuple has a centred equivariant polynomial Cartan witness and a local algebraic, hence also analytic, realisation. The additional rows test only the given lattice, its labelled integer values, and rational, mod-two or mod-three spans. With their dimensions bounded, the cost of each such numerical test is polynomial in the binary input length.

2.1.1 The single-curve classification

For one primitive curve class c, the complete support is $\{c, 2c, \dots, \ell c\}$, where $1 \leq \ell \leq 6$. Brown–Wemyss [10, Theorems 1.3–1.4 and Corollary 1.6 (9.11)] give the following entire numerical list:

ℓ	$(n_c, n_{2c}, \dots, n_{\ell c})$
1	(a), $\quad a \geq 1$
2	(a, b), $\quad b \geq 1$, $\quad a \in \{4, 6, \dots, 2b+2\} \cup \{2b+3\}$
3	$(6, 3, 1)$, $\quad$ or $(7, b, 1)$, $\ b \geq 4$
4	$(6, 5, 2, 1)$
5	$(8, 6, 4, 2, 1)$
6	$(6, 6, 4, 3, 2, 1)$.

At length three the unit multiplicity lies on $3c$. Thus a unit multiplicity can occur on a nonprimitive label.

2.2 Root orders and the multiplicities of curve classes

Fix a Du Val section with irreducible ADE root system Φ, set I of internal simple vertices and nonempty complement J. Write $\pi_I : Q(\Phi) \to Q(\Phi)/Q(\Phi_I)$ for the quotient map. The curve basis is given by the images of the simple roots indexed by J. After the base change $t = s^m$, let $\lambda(s)$ denote the lift of the classifying arc to the complex Cartan space of Φ. Put

$$R_\beta = \{\alpha \in \Phi^+ : \pi_I \alpha = \beta\}, \quad c = \prod_K c_K, \quad m = \operatorname{lcm}_K h_K,$$

where K runs over internal connected components and c_K is a Coxeter element of order h_K. The empty internal system has $c = 1, m = 1$. The local simultaneous partial resolution of Katz–Morrison [4, §6, Theorem 3] and the discriminant formula of Nabijou–Wemyss [5, (3.A)] give

$$\lambda(\zeta_m s) = c\lambda(s), \qquad n_\beta = \frac{1}{m} \sum_{\alpha \in R_\beta} \operatorname{ord}_s \alpha(\lambda(s)). \tag{2.1}$$

The lift satisfies $\lambda(0) = 0$, and every external root has positive finite order.

Section 2.2.1 illustrates the fibre formula on a two-curve partial resolution.

2.2.1 Three divisors on a two-curve partial resolution

Take the chain 1–2–3 of type A_3 and the partial resolution whose curves correspond to $J = \{1, 3\}$. In their ordered basis the labels are $10 = (1, 0)$, $01 = (0, 1)$ and $11 = (1, 1)$. In simple-root coordinates $z_i = \alpha_i(\lambda)$ the divisor equations pull back to

$$P_{10} = z_1(z_1 + z_2), \qquad P_{01} = z_3(z_2 + z_3), \qquad P_{11} = z_1 + z_2 + z_3.$$

These are the three polynomials used to compute the algebra and cocenter dimensions in [7, §4.3.7]. For example, the two roots α_1 and $\alpha_1 + \alpha_2$ both project to 10, which accounts for the two factors of P_{10}.

We now determine their possible intersection orders in this presentation. The internal reflection exchanges the factors of P_{10} and those of P_{01}. On $t = s^2$ put $U = z_2/2$, $H_1 = z_1 + U$ and $H_3 = z_3 + U$. The two H_i are invariant and hence polynomials or series in t with zero constant term. The series U is odd in s. The smoothing parameter at the internal A_1 point is a nonzero multiple of U^2. Its derivative in t must be nonzero for the total space to be smooth, so $\operatorname{ord}_s U = 1$. Thus

$$P_{10} = H_1^2 - U^2, \qquad P_{01} = H_3^2 - U^2, \qquad P_{11} = H_1 + H_3.$$

Both squares H_i^2 vanish to order at least two in t, while U^2 has order one in t. This forces $n_{10} = n_{01} = 1$. Conversely, given any $k \geq 1$, choose $U = s$, $H_1 = 0$, $H_3 = t^k$. The three evaluations are $-t$, $t^{2k} - t$ and t^k, so this single smooth arc gives precisely $(1, 1, k)$.

The full A_2 resolution has the same three labels. There the arc $(t^2, -t^2 + t^3)$ gives $(2, 2, 3)$. It follows that the three equations above give the whole answer for the chosen A_3 presentation and only part of the answer for this support. This is the reason for taking a union over compatible presentations in the necessity proof.

2.2.2 Smoothness and the reduced divisor equation

For each internal component K, the exact condition is $\operatorname{ord}_s \alpha(\lambda) = m/h_K$ for every internal root of K. Near the corresponding Du Val point the total space is smooth exactly when the constant smoothing coordinate has nonzero first derivative in t [4, §6, Theorem 3]. In homogeneous basic invariants this is the top-degree invariant, up to a nonzero scalar and terms of order at least two along a centred arc. Write the invariants as $b_j(t) = ta_j(t)$ and their degrees as d_j, with $h = h_K$. For $t = z^h$ a lift $zv(z)$ must satisfy $p_j(v(z)) = z^{h-d_j} a_j(z^h)$. Scale a regular primitive Coxeter eigenvector to the specified nonzero top value. The lower invariants vanish on this vector and the Chevalley map is étale there. The local inverse yields v, and uniqueness gives Coxeter equivariance. All internal roots have order one in z. Conversely, the specified internal orders and equivariance give a regular primitive leading coefficient. Its lower invariants vanish and its top invariant is nonzero, since the zero fibre of the finite Chevalley quotient is $\{0\}$. This gives the required first derivative.

The central partial surface is already smooth elsewhere. Finite external orders ensure that after shrinking the germ its punctured disc misses the discriminant; the projective partial resolution therefore restricts to a small contraction to an isolated threefold singularity with a Du Val central hyperplane section. Its Gorenstein target and smallness give crepancy. A polynomial equivariant lift has invariant coordinates in $\mathbb{C}[t]$ and gives a local algebraic realisation.

The fibre formula in (2.1) is the order of the product of the root forms defining the appropriate curve divisor. Indeed each nonzero fibre is one W_I-orbit [5, Lemma 2.3, Theorem 2.6 and (3.A)]. At a generic point of an external hyperplane H_α the only possible reflection inertia is generated by $s_\alpha \notin W_I$. Its coefficient in the pullback of the reduced curve divisor is therefore one. Distinct roots over a fixed nonzero positive label give distinct hyperplanes; pulling their product back by $t = s^m$ gives the factor $1/m$.

2.2.3 Prescribing the orders of linear forms

For a centred arc, a root form has order k when the coefficients of degrees $1, \dots, k-1$ vanish and its coefficient of degree k is nonzero. We say that the root exits at degree k. It survives through a degree if all its coefficients through that degree vanish. Suppose one coefficient vector has to meet several such prescriptions. A form in the span of the forms required to vanish will also vanish. The next lemma gives the converse, and makes all the required nonvanishings hold for a single vector.

Lemma 2.2. *Let V be a finite-dimensional complex vector space with a specified finite collection of nonzero linear forms α, and prescribe positive integral orders d_α. A polynomial arc has exactly these orders if and only if, for each attained level k,*

$$\alpha \notin \operatorname{span}_{\mathbb{C}}\{\gamma : d_\gamma > k\} \qquad (d_\alpha = k).$$

For equivariant arcs let $E_k = \ker(c - \zeta_m^k)$ in V. The test is

$$\alpha|_{E_k} \notin \operatorname{span}_{\mathbb{C}}\{\gamma|_{E_k} : d_\gamma > k\} \qquad (d_\alpha = k). \tag{2.2}$$

For a prefix through degree H, a form assigned ∞ vanishes through H, and only attained finite levels $k \le H$ are tested. For rational forms the ordinary span test can be performed over $\mathbb{Q}$.

Proof. At degree k the coefficient must annihilate every higher-order form and avoid the kernels of all forms that exit at k. Each latter kernel is proper in the permitted coefficient space exactly when the displayed condition holds. A finite union of proper complex hyperplanes cannot cover that space. Choose a coefficient outside this union for every attained level and put all other coefficients equal to zero. This is a polynomial witness. Necessity follows by evaluating its coefficient at the first nonzero degree of each form. □

The generic choice is effective. If the permitted coefficient space has basis $w_0, \dots, w_{d-1}$ and at most M nonzero forms must be avoided, try $\sum_{j=0}^{d-1} T^j w_j$ for integers $0 \le T \le M(d-1)$. Every forbidden form gives a nonzero polynomial of degree at most $d-1$, so at least one trial succeeds. The calculations take place in the rational or explicitly specified cyclotomic field. The successful trial gives the required coefficient vector.

The ordinary root-contact criterion used throughout means precisely this rational span condition on the given root labels. It is equivalent to requiring every strict superlevel set to be a root flat. This is the elementary nonemptiness argument underlying root-valuation strata [8].

2.2.4 One surviving orbit in a root fibre

With m and the h_K as in (2.1), put $L = \max_K(m/h_K)$, with $L = 0$ if I is empty.

Write $\nu_\alpha = \operatorname{ord}_s \alpha(\lambda)$, and denote the truncation of a smooth admissible arc through degree L by τ. Let Ψ be the roots annihilating that truncation. It is a c-stable root flat, disjoint from internal roots. Define the exited contribution and surviving length by

$$b_{\tau,\beta} = \frac{1}{m} \sum_{\alpha \in R_\beta \setminus \Psi} \nu_\alpha, \qquad \ell_{\tau,\beta} = |R_\beta \cap \Psi^+|.$$

The background is integral: its root product is c-invariant, so its first exponent is divisible by m.

Lemma 2.3. *Every nonempty $R_\beta \cap \Psi^+$ is one c-orbit.*

Proof. Let $A = m^{-1} \sum c^j$. Its kernel is $\operatorname{span} I$. Two survivors with the same label have the same nonzero average. If they had different orbits, all pairings $(\alpha, c^j \alpha')$ would be nonpositive: a positive pairing would produce the internal surviving root $\alpha - c^j \alpha'$. Their average would then be nonpositive, contrary to $\|A\alpha\|^2 > 0$. □

Thus on a surviving fibre its only possible common order is

$$\nu_\alpha = \frac{m}{\ell_{\tau,\beta}} (n_\beta - b_{\tau,\beta}). \tag{2.3}$$

Test the finitely many zero patterns of the prefix by Lemma 2.2 through degree L; each passing prefix extends by generic higher coefficients over one more full period.

2.2.5 Restriction to a saturated flat

The enhanced arrangement and integral flop comparison of [5, §§4–5] give a restriction operation on complete spectra. If a complete spectrum is realisable and $U \subseteq \Lambda_{\mathbb{Q}}$ is spanned by its signed labels, its restriction to $\Lambda \cap U$ is, after a chamber change, a direct sum of smooth-source local spectra, one for each connected contracted fibre. Indeed choose a generic rational point in $U^\perp$ and a nef chamber incident to its arrangement face. The corresponding smooth crepant resolution has a relatively semiample face contraction. Its contracted curve classes form a subset of the chamber basis and span precisely the saturated lattice $\Lambda \cap U$. This contraction is small and crepant; each connected exceptional fibre has an isolated cDV image. GV counts supported on their disjoint local neighbourhoods split, and the integral flop formula identifies them with the original values. In particular, restriction to a primitive supported ray gives a smooth simple flop.

2.2.6 Reduced root supports

For a full irreducible reduced crystallographic support R^+ put $e = 1$ in ADE, $e = 2$ in B, C, F_4, and $e = 3$ in G_2. In ADE use $v = n$ and only rational spans. Otherwise use

$$Q = \mathbb{Z}R, \quad D = Q/\mathbb{Z}R_{\text{long}}, \qquad v_\beta = \begin{cases} n_\beta - (e-1), & \beta \text{ short}, \\ e n_\beta, & \beta \text{ long}. \end{cases}$$

All v_β must be positive. For $H_k = \{\gamma : v_\gamma > k\}$ the test at each attained level is

$$\begin{cases} \beta \notin \operatorname{span}_{\mathbb{Q}} H_k, & e \mid k, \\ \bar{\beta} \notin \operatorname{span}_{\mathbb{F}_e} \overline{H_k} \subset D, & e \nmid k, \end{cases} \qquad (v_\beta = k). \tag{2.4}$$

The character calculation and the comparison of partial resolutions give this test also when some multiplicities are one or two.

The diagram folds giving the two character spaces are $D_{r+1} \to B_r$, $A_{2r-1} \to C_r$, $E_6 \to F_4$, and $D_4 \to G_2$. The fixed-character forms have the rational root matroid. The moving-character forms are respectively a single nonzero class, the A_{r-1} graphic forms, the three A_2 forms, and a single nonzero class over the Eisenstein integers. Reduction gives exactly the subset ranks in D, of dimensions $1, r-1, 2, 1$. For each of these configurations the two representations have the same subset ranks. Lemma 2.2 gives the test.

For full ADE the roots themselves give the orders. In type B remove one generic active terminal in the classical model. In type C use paired scalar coordinates with a generic common odd linear term. In both cases the short fibres have background one. The low-rank classical overlaps are listed in Appendix B.3.3.

The short-fibre backgrounds in G_2 and F_4 are two and one, respectively. Appendix B.3.2 gives the primitive coefficients, the root counts and the construction of the folded arc.

2.3 Marked supports and root orders

Two presentations with the same signed support $B^{\pm} = B \cup (-B)$ can give different multiplicities. We first determine the image of a fixed presentation, then take the union in the given marking, extending n to $B^{\pm}$ by sign.

2.3.1 The lattice and the chamber

For an irreducible ADE root system Φ, let I be a proper subset of a simple system, Φ_I its root subsystem, and $\pi_I : Q(\Phi) \longrightarrow \Lambda_I = Q(\Phi)/Q(\Phi_I)$ the quotient of root lattices. Its signed support is $B_I^{\pm} = \pi_I(\Phi) \setminus \{0\}$. A regular functional on Λ_I selects one sign of each pair. An incident Weyl chamber gives the basis of Λ_I formed by the images of its nonvanishing simple roots. Indeed, the pullback of the functional vanishes precisely on Φ_I. In a Weyl chamber whose closure contains it, the vanishing simple roots form a basis of $Q(\Phi_I)$. Every positive projected root is a nonnegative integral combination of the other simple-root images, and each image occurs. Thus these images are exactly the indecomposable elements of the positive-label monoid. This applies to every chamber of the signed support.

Proposition 2.4. *All identifications of a model signed support with $B^{\pm}$ in its ordered chamber marking can be found by a finite test. An isometry of ambient root systems carrying the internal subsystem onto the new internal subsystem transports the numerical image, with values taken at the positive representatives in the transported chamber.*

Proof. For any spanning signed set D, any lattice isomorphism T carrying D onto $B^{\pm}$ preserves the inverse moment metric, since $M_B = T M_D T^{\mathsf{T}}$ for $M_D = \frac{1}{2}\sum_{u \in D} uu^{\mathsf{T}}$. Fixing r linearly independent vectors in D as the columns of a matrix U, any valid map T is uniquely determined by the image matrix $V = TU$ of columns drawn from the finite set $B^{\pm}$ with matching Gram matrix $V^{\mathsf{T}} M_B^{-1} V = U^{\mathsf{T}} M_D^{-1} U$. Checking integrality of $T = VU^{-1}$ in the lattice bases, $|\det T| = 1$, $T(D) = B^{\pm}$, and that the ordered chamber generators of $T^{-1}B$ map to $(c_1, \dots, c_r)$ therefore isolates all admissible identifications in finitely many steps.

For transport, an ambient root-system isometry carrying one internal subsystem onto another maps external root fibres and internal roots bijectively. Conjugating the transported component Coxeter elements to the chosen ones by an element of the new internal Weyl group acts trivially on the quotient lattice, so the composite transports every smooth Cartan arc while preserving all root orders and labels. □

2.3.2 A common coefficient in every degree

Keep the Cartan space, external root fibres R_β, Coxeter product c, cover $t = s^m$, and last mandatory internal degree $L = \max_K(m/h_K)$ (0 if I is empty) of Section 2.2. Put $d_K = m/h_K$ and $E_k = \ker(c - \zeta_m^k)$. Lemma 2.2 tests the exiting forms on E_k against the span of the deeper forms, and Lemmas 2.3 and 4.2 give the two orbit statements for the surviving and full fibres, respectively.

A prefix through L is a polynomial $\sum_{k=1}^{L} v_k s^k$ with $v_k \in E_k$. Its order pattern is admissible if it occurs with the required internal orders.

Proposition 2.5. *Let e_α be constant on signs and c-orbits, with values in $\{1, \dots, L, \infty\}$. This pattern is admissible precisely when its internal assignments are d_K and it passes Lemma 2.2 through degree L. For an admissible assignment let Ψ be the roots assigned ∞ and put*

$$b_\beta = \frac{1}{m} \sum_{\alpha \in R_\beta \setminus \Psi} e_\alpha, \qquad \ell_\beta = |R_\beta \cap \Psi|.$$

Then b_β is an integer. The positive integral tuple n occurs with this prefix exactly when the following hold for $\beta \in B$.

1. *For $\ell_\beta = 0$ one has $n_\beta = b_\beta$.*
2. *For $\ell_\beta > 0$ every root in $R_\beta \cap \Psi$ is assigned the depth*

$$a_\alpha = \frac{m}{\ell_\beta}(n_\beta - b_\beta) > L. \tag{2.5}$$

These depths pass (2.2) amongst all survivors simultaneously, at every attained degree above L.

Proof. Lemma 2.2 constructs the prefix. On a c-orbit of length q the orders agree, and each evaluation is invariant under $s \mapsto \zeta_m^q s$. Its finite order is divisible by m/q. Thus each exited orbit contributes an integer to b_β. Lemma 2.3 leaves at most one orbit per fibre, so (2.1) forces (2.5). Here $\ell_\beta \mid m$ whenever $\ell_\beta > 0$. Conversely, append the coefficients supplied by Lemma 2.2 for the survivor depths, using signs to include the negative fibres. Earlier exits, including those of all internal roots, keep their orders. The resulting polynomial realises (2.1). Any admissible prefix also extends before n is specified. Take a generic coefficient in each of the next m degrees. Every surviving root is nonzero on at least one of their character spaces. □

Corollary 2.6. *The intrinsic image on B is the union of the images in Proposition 2.5 over all compatible presentations, their finitely many admissible order patterns through L, and the integral identifications of Proposition 2.4.*

Proof. Truncate the lift of a realising contraction at L. Its remaining orders satisfy (2.5) and (2.2). In the reverse direction each passing prefix and tail give one polynomial lift to which the geometric input applies. Thus a proposed numerical row is exact once every compatible image is contained in it and every tuple of the row has a witness in some compatible image. A proved inclusion absorbs a smaller image, and disjoint images give separate alternatives. To use a single formula in every permissible frame one checks its invariance under the corresponding signed lattice maps. □

2.3.3 The integer conditions for a fixed presentation

Fix a presentation $p = (\Phi, I)$ and its quotient basis. We now express its image as a condition $Q_p(n)$ on the integers n_β. Proposition 2.5 determines the depth of each surviving orbit once

the orders through L are known. We must find all the initial patterns and test the resulting depths, however large the n_β are.

For a divisor d of m, let $\Phi_d(X)$ be the dth cyclotomic polynomial and put $U_d = \ker \Phi_d(c)$ in the rational Cartan space. A root a acts there by the form $v \mapsto a^{\mathsf{T}} G v$, where G is the Cartan matrix in the simple-root basis.

Lemma 2.7. *Let $d = m/\gcd(k, m)$ and let H be a c-stable set of rational root forms. For any rational root form a,*

$$a|_{E_k} \in \operatorname{span}_{\mathbb{C}}(H|_{E_k}) \quad \Longleftrightarrow \quad a|_{U_d} \in \operatorname{span}_{\mathbb{Q}}(H|_{U_d}).$$

Proof. Over $\mathbb{Q}(\zeta_m)$, the space U_d is the direct sum of the eigenspaces for all primitive characters of order d. The span of $H|_{U_d}$ is rational and c-stable, so its scalar extension splits under the same spectral projections in the dual space. Containment on E_k is a rank equality over $\mathbb{Q}(\zeta_m)$. Galois conjugation gives the same equality on every primitive order-d eigenspace. Taking the direct sum and descending the rank equality to $\mathbb{Q}$ proves the forward implication. Restriction gives the converse. On E_k the forms $a, ca, c^2a, \ldots$ are scalar multiples of one another. On U_d we take the rational span of the entire orbit. □

Orders through the last internal degree. Suppose S is the set of roots still vanishing before degree k. For $Z \subseteq S$, the allowed coefficient space is $E_k \cap \bigcap_{a \in Z} \ker a$. By Lemma 2.2, a single vector in this space is nonzero on every root in $S \setminus Z$ precisely when Z is a flat of the restricted forms. In other words, Z contains exactly the roots of S whose restrictions lie in $\operatorname{span}_{\mathbb{C}}(Z|_{E_k})$.

Start with all roots, grouped by sign and by c-orbit. At degrees $k = 1, \ldots, L$, form the zero span on $U_{m/\gcd(k,m)}$ and successively adjoin the spans of whole surviving orbits. Closing each resulting span gives one possible Z by Lemma 2.7. Require every internal root in Φ_K to stay in Z for $k < d_K$ and to exit at $k = d_K$. Give the roots in $S \setminus Z$ order k and continue with Z. This finite induction enumerates every smooth initial pattern: an actual arc gives one of the flats at each step, and coefficient avoidance realises every admissible sequence of flats. Powers of the individual c_K commute with c and preserve the quotient labels, so one pattern per orbit under these rotations suffices.

Orders above that degree. For a fixed pattern, impose $n_\beta = b_\beta$ on each fibre with no survivor. Index the other fibres by i. Each contains one surviving orbit, of length ℓ_i, with forced depth $k_i = (m/\ell_i)(n_{\beta_i} - b_{\beta_i}) > L$ by (2.5). For each $d \mid m$, let $\mathcal{F}_d$ be the closed sets of surviving orbit indices obtained from their rational spans on U_d. For $d = m/\gcd(k_i, m)$, the condition for orbit i to exit is

$$\bigvee_{\substack{F \in \mathcal{F}_d \\ i \notin F}} \bigwedge_{\substack{h \notin F \\ h \neq i}} (k_h \leq k_i). \tag{2.6}$$

Indeed a flat containing all strictly deeper orbits and excluding i exists precisely when the closure of those deeper orbits excludes i. An identically zero form lies in every flat. Several orbits may exit at the same depth by one application of Lemma 2.2.

For each initial pattern impose the empty-fibre equalities, the inequalities $k_i > L$, and (2.6) for every surviving orbit. Take the disjunction of these conjunctions over the initial patterns. This defines $Q_p(n)$ by a finite formula of linear integer arithmetic with congruences. The residue of k_i modulo m determines d and each ℓ_i divides m. A smooth arc supplies a true disjunct. Conversely, choose the initial coefficients of a true disjunct and one vector at each attained depth k_i as in the proof of Lemma 2.2. Their finite sum is a centred equivariant polynomial. It has all the prescribed orders at once and gives the required contraction by Section 2.2.2.

3 Classical supports and boundary factorisation

Having established the Cartan–GV interface, we now resolve all classical supports in arbitrary curve rank. How can one decide directly from the marked integers whether a classical spectrum arises from a smooth-source arc, without searching over ambient presentations? We first reduce every classical presentation to singleton and quadratic blocks and recover the canonical frame intrinsically (Sections 3.1–3.2), then prove the canonical boundary factorisation and sharp three-block integer criterion on full BC_r (Theorems 3.1 and 3.3), and finally extend the test to mixed singleton–quadratic frames (Proposition 3.5).

3.1 Reduction of the classical presentations

For a type-D presentation each permutation block is a smooth plane branch C_i, with nonsingletons written $t = \phi_i(X)$, $\operatorname{ord}_X \phi_i \geq 2$, and singletons written $X = x_i(t)$. With $\iota(X,t) = (-X,t)$, signed pair values are the intersections $I(C_i, C_j)$ and $I(C_i, \iota C_j)$, and a doubled label satisfies

$$I(C_i, \iota C_i) = 1 + 2n_{2u_i}.$$

Terminal contributions are intersections with an even long branch and a short scheme $X^2 = Z(t)$, where Z has odd order or is zero. Apply the single common coordinate change $T = t + aX^2$ for generic a. It commutes with ι, preserves every intersection number, makes every nonsingleton quadratic, and preserves every singleton graph by the implicit function theorem. The long terminal becomes active. The short square is solved from $Y = Z(T - aY)$; its odd order is preserved, and an active leading coefficient c changes to $c/(1+ac)$. Thus distinct active terminal squares remain distinct. The new terminal is D_2 and its internal roots have primitive order. The quotient lattice and singleton flags are unchanged. The same change of coordinates works in every classical rank. Type A uses a common generic transverse projection of smooth branches and reduces to ordinary A_r contacts. The usual low-rank identifications are $D_3 = A_3$, the full A_1 cases, and the mixed two-block A_2 triple.

3.2 Recovering the frame from the support

For $r \geq 4$, recover the frame from the simple-sum graph: vertices are the chamber generators c_i, and ij is an edge exactly when $c_i + c_j \in B$.

There are three frames, with M read from membership of $2u_i$. The terminal frame has $c_i = u_i - u_{i+1}$ for $i < r$ and $c_r = u_r$, and its signed support is

$$\{\pm u_i \pm u_j : i < j\} \ \cup \ \{\pm u_i\} \ \cup \ \{\pm 2u_i : i \in M\}.$$

Its lattice is $\mathbb{Z}^r$. The nonterminal frame has signed support

$$\{\pm u_i \pm u_j : i < j\} \cup \{\pm 2u_i : i \in M\}$$

in the even-sum lattice. If $r \in M$, its last generator is $2u_r$; otherwise its last generator is $u_{r-1} + u_r$. All preceding generators are $u_i - u_{i+1}$. Full A_r is recognised separately as the interval root support in its chamber basis.

In a path graph the terminal frame is distinguished and oriented by the unique doubled height-three edge pattern $c_{r-1} + 2c_r$. The nonterminal path ending in $2u_r$ is distinguished by the unique three-vertex pattern $c_{r-2} + 2c_{r-1} + c_r$. These tests are followed by exact comparison with the whole displayed support. In the other nonterminal frame the graph is a D_r fork if $r - 1 \notin M$, and a triangle with one attached path if $r - 1 \in M$. The path attached to a triangle specifies its distinguished vertex, also in rank four. A fork in rank $r \geq 5$ has a unique long arm. In both cases exchanging the two short ends changes the sign of u_r. In a rank-four

fork the optional flags are $2u_1 = (2, 2, 1, 1)$ and $2u_2 = (0, 2, 1, 1)$. Either distinguishes c_1 from the other two leaves. With neither flag one obtains full D_4, with its triality. Thus the frame and all flags are recovered up to symmetries that leave the numerical criterion unchanged.

No exceptional partial presentation has one of these classical supports in rank at least four. All classical chamber coordinates are at most two, whereas the highest-root marks in the exceptional numbering are

$$E_6 : (1, 2, 3, 2, 1, 2), \quad E_7 : (2, 3, 4, 3, 2, 1, 2), \quad E_8 : (2, 4, 6, 5, 4, 3, 2, 3).$$

These exclude E_8 already in rank three. In E_7 only one rank-four choice remains; its simple-sum triangle has a short end of maximum coefficient two, incompatible with a classical fork frame. In E_6 the possible rank-five graph has paths at two different triangle vertices. In rank four, the possible triangle graphs have the same short-end obstruction; the remaining path has two distinct $(1, 2, 1)$ patterns instead of one. These direct graph tests exclude all cases. The shear therefore gives the numerical image for every classical support in these ranks.

For ranks two and three use the explicit bounded frame identifications: full A_2, $B_2 = C_2$, the two five-label chamber types, full BC_2; and in rank three $D_3 = A_3$, the nonterminal partial-doubling frames, and the eight terminal frames. For example, in D_4 number the central node 2 and the leaves $1, 3, 4$, with $3, 4$ the two fork ends. The internal pair $\{3, 4\}$ gives two singletons and a terminal D_2. The pair $\{1, 3\}$ gives two quadratic blocks. Exchanging leaves $1, 4$ carries one pair to the other and identifies their marked $B_2 = C_2$ tests. A permutation of at most three chamber generators resolves the low-rank ambiguity, followed by exact support comparison. The exceptional five-label comparison is proved directly in Section 6.3.4; the remaining low-rank scalar comparisons are specified in Appendix B.3. An isolated two-label $D_2 = A_1 + A_1$ support is excluded: it is not an irreducible one-point cDV contraction.

3.3 Contact classes on full BC_r

Let B be the full positive BC_r support. Its primitive doubled rays recover the short labels $u_1, \dots, u_r$ intrinsically. Extend the spectrum by sign symmetry, and write

$$a_i = n_{u_i}, \qquad b_i = n_{2u_i}, \qquad p_{ij} = n_{u_i - u_j}, \qquad q_{ij} = n_{u_i + u_j}.$$

The pair values determine which quadratic blocks have the same leading square. A class with $a_i > 4$ must share its leading square with one of the two terminal coordinates. These are the two possible anchors in the scalar construction. The integer conditions below determine how many such classes occur, including when $b_i = 1$ or $p_{ij} = q_{ij} = 2$.

One uniform rank test. For a finite configuration R with assigned positive depths d_α, define the two-character test as follows. At an attained even depth k, require

$$\alpha \notin \operatorname{span}_{\mathbb{Q}}\{\gamma : d_\gamma > k\} \quad (d_\alpha = k).$$

At an attained odd depth use the same condition over $\mathbb{F}_2$ on the following residue vectors:

$$u_i \pm u_j \longmapsto e_i + e_j, \qquad 2u_i \longmapsto 0,$$

and, when the roots u_i are present, $u_i \mapsto e_i$. For C_m this is its root lattice modulo the long-root lattice; for BC_m it is $\mathbb{Z}^m/2\mathbb{Z}^m$. The $m = 1$ C configuration has only $2u_1$ and zero odd residue, so its positive even depth is unrestricted.

Theorem 3.1 (Canonical boundary factorisation for BC_r)**.** *A positive full-BC_r tuple is realised by a smooth-source flop if and only if all the following tests pass.*

1. *One has* $a_i \geq 4$ *and* $p_{ij}, q_{ij} \geq 2$.

2. $p_{ij} > 2$ *if and only if* $q_{ij} > 2$, *and*

$$i \sim j \quad \Longleftrightarrow \quad i = j \text{ or } p_{ij} > 2$$

is an equivalence relation. Call its equivalence classes the contact classes.

3. *Every class* C *has either* $a_i = 4$ *for all* $i \in C$, *or* $a_i > 4$ *for all* $i \in C$. *Call the latter classes attached. There are at most two attached classes.*

4. *For each unattached class* C, *apply the two-character test to its* $C_{|C|}$ *configuration with depths*

$$d_{u_i - u_j} = p_{ij} - 1, \quad d_{u_i + u_j} = q_{ij} - 1, \quad d_{2u_i} = 2b_i.$$

For each attached class, apply the same test to its full $BC_{|C|}$ *configuration with the additional depths*

$$d_{u_i} = a_i - 3.$$

The partition, attached flags, and depths are read directly from the numerical input. Distinct classes have exactly $p_{ij} = q_{ij} = 2$*; after this forced boundary interaction, their rank tests are independent.*

Proof of Theorem 3.1. Every full-BC_r spectrum has a model with r quadratic blocks and terminal D_2. This follows from the classical shear, the two exceptional comparisons in rank two (Appendix B.3.3), and the length-two classification in Section 2.1.1.

On $t = s^2$ write a quadratic pair as $x_i^{\pm} = A_i(s^2) \pm sB_i(s^2)$ with $B_i(0) \neq 0$, and let Y, Z be the odd terminal coordinates. Their sums and differences have order one. At least one terminal coordinate is active. If the other is inactive, its interaction with every quadratic block contributes two; it can be replaced by a generic active odd coordinate without changing any multiplicity. We may therefore assume that Y, Z have nonzero leading coefficients with distinct squares.

If $B_i(0)^2 \neq B_j(0)^2$, all four differences and all four sums between the two blocks have order one, and both multiplicities are two. If their leading squares agree, two orders in each fibre are at least two and the other two are one, so both multiplicities exceed two. Thus the contact classes are exactly the equal-leading-square classes. A class matches either no terminal square, giving $a_i = 4$ throughout the class, or exactly one, giving $a_i > 4$ throughout. There are only two terminal squares, hence at most two attached classes.

Align signs inside one contact class. For an unattached class subtract its common leading odd term and set $F_i^+ = x_i^+ - As$, $F_i^- = -x_i^- - As$. For an attached class subtract the entire matching odd terminal coordinate instead. In either case

$$F_i^-(s) = -F_i^+(-s),$$

and the prescribed pair/self depths are exactly

$$\operatorname{ord}(F_i^+ - F_j^+) = p_{ij} - 1, \quad \operatorname{ord}(F_i^+ - F_j^-) = q_{ij} - 1, \quad \operatorname{ord}(F_i^+ - F_i^-) = 2b_i.$$

In the attached case also $\operatorname{ord} F_i^{\pm} = a_i - 3$. All of these depths are at least two.

At even degree, the coefficient values on a paired label are $(z_i, -z_i)$; the resulting forms are exactly the rational C_m or BC_m vectors. At odd degree the paired values agree, so the pair forms are the edges of a complete graph. The attached case adds an origin vertex, whose edges have forms z_i. A graphic matroid has the same dependence relations over $\mathbb{C}$ and $\mathbb{F}_2$, represented by the residue vectors stated above. At each attained depth, a coefficient exists precisely when every form required to become nonzero is outside the span of the forms still

required to vanish. Generic coefficient choice outside finitely many proper hyperplanes proves both necessity and sufficiency of the two-character tests and constructs polynomial F's.

For global sufficiency choose $Y = s, Z = 2s$. Assign the at most two attached classes distinct leading coefficients $1, 2$ and choose distinct positive integral coefficients at least three for all unattached classes. For a class with coefficient A_C, put

$$x_i^+ = A_C s + F_i^+, \qquad x_i^- = -A_C s - F_i^-.$$

These form a quadratic pair under $s \mapsto -s$. Its internal difference has order one. Between classes every signed fibre has order sum four, giving $p = q = 2$. Within a class the fibre sum is one plus its prescribed paired depth; the diagonal gives b_i. For an attached class, the matching terminal contributes $1 + d_{u_i}$ and the other contributes two, so $a_i = 3 + d_{u_i}$. An unattached class contributes two to each terminal, so $a_i = 4$. Finally $Y \pm Z$ have order one. This constructs a smooth Cartan witness with exactly the complete tuple and proves sufficiency. □

Corollary 3.2. *Among realisable full-BC_r tuples, the global one-anchor BC_r two-character test with depths $a_i - 3$, $2b_i$, $p_{ij} - 1$, $q_{ij} - 1$ holds exactly when there is at most one attached contact class. The additional two-anchor sector consists precisely of tuples with two attached classes; each of these classes passes an independent one-anchor rank test, and all interactions between the classes are $p = q = 2$.*

Proof. For scalar paired labels and one fixed origin, a contact $a_i - 3 > 1$ means the corresponding first odd coefficient equals that of the origin. Thus all such labels lie in one contact class. Conversely, placing one attached class at the origin and assigning distinct nonzero first odd coefficients to all remaining classes glues the independent constructions in the proof of Theorem 3.1 into a single one-anchor contact model. □

Section 3.3.1 gives a tuple requiring both terminal coordinates.

3.3.1 Two attached classes in BC_2

For

$$(p, q, b_1, b_2, a_1, a_2) = (2, 2, 3, 4, 9, 11)$$

there are two singleton contact classes, both attached. Their local BC_1 depths are respectively $(6, 6)$ and $(8, 8)$ on $(u, 2u)$, so both rank tests pass. An explicit D_6 polynomial witness is

$$(x_1^+, x_1^-, x_2^+, x_2^-, Y, Z) = (s + s^6, -s + s^6, 2s + s^8, -2s + s^8, s, 2s).$$

Internal roots have order one; diagonal roots have orders six and eight; all cross-block roots have order one. The terminal fibre sums give 9 and 11. This tuple fails the global one-anchor test at odd level one: the residue of $u_1 - u_2$ lies in the span of the higher-depth residues u_1, u_2. The canonical class factorisation explains exactly why its actual two-anchor realisation is nevertheless valid.

3.3.2 Exceptional sections with full BC_2 support

Two exceptional presentations have the full classical BC_2 support. In the one-based numbering of Appendix B.2 they are $E_7/15$ and $E_8/17$. Why do they give no additional spectra? After the first exits, the surviving scalar system is A_5 in E_7 and A_4 in E_8. On $z = s^{m/2}$ there are two pairs of coordinates $F_i^+(z), F_i^-(z) = -F_i^+(-z)$ over $\pm u_i$. There are also odd zero-weight coordinates Y_1, Y_2 in E_7, and Y_1 in E_8. The respective initial contributions on the axes, pair labels and doubles are $(2, 1, 0)$ and $(3, 1, 0)$. These fibre and action data come from projecting the roots as in Appendix B.3.3.

In E_7 the difference $Y_1 - Y_2$ is an internal root of z-order one. In E_8 adjoin $Y_2 = cz$ with c chosen so that $Y_1 - Y_2$ and both $F_i^+ - Y_2$ have order one. In either case take

$$x_i^+(z) = Mz + F_i^+(z), \qquad x_i^-(z) = x_i^+(-z), \qquad T_j(z) = Mz + Y_j(z).$$

Choose M outside the finitely many values for which an unmatched block or terminal interaction loses its linear term. Then $T_1 + T_2$ and $T_1 - T_2$ have order one. In the classical model the pair values are $1 + \operatorname{ord}_z(F_i^+ - F_j^\pm)$, the doubles have value $\frac{1}{2}\operatorname{ord}_z(F_i^+ - F_i^-)$, and the axes have value $2 + \sum_{j=1}^{2} \operatorname{ord}_z(F_i^+ - Y_j)$. For E_8 the added coordinate supplies the third unit on each axis. These are precisely the original exceptional values. Thus both exceptional images lie in the image of the classical two-terminal model.

3.4 The integer test on three blocks

Eliminating the paired depths in Theorem 3.1 gives five integer conditions. Each involves at most three coordinate blocks.

Theorem 3.3 (Integer criterion for all full-BC multiplicities)**.** *With the notation above, put $\ell_{ij} = \min(p_{ij}, q_{ij})$ and $z_{ij} = \min(b_i, b_j)$. A positive tuple is realisable if and only if:*

1. *For every i,*
$$\begin{cases} 4 \le a_i \le 2b_i + 2, & a_i \text{ even}, \\ a_i = 2b_i + 3, & a_i \text{ odd}. \end{cases}$$
2. *For every pair i, j, $\ell_{ij} \ge 2$, and*
$$\begin{cases} p_{ij} = q_{ij} = \ell_{ij} \le 2z_{ij}, & \ell_{ij} \text{ even}, \\ \ell_{ij} = 2z_{ij} + 1, \quad (p_{ij} \ne q_{ij} \Rightarrow b_i = b_j), & \ell_{ij} \text{ odd}. \end{cases}$$
3. *If $\ell_{ij} > 2$, then $a_i = 4$ if and only if $a_j = 4$. If both exceed four, additionally*
$$\begin{cases} p_{ij} = q_{ij} = \min(a_i, a_j) - 2, & a_i \ne a_j, \\ \ell_{ij} \ge a_i - 2, & a_i = a_j. \end{cases}$$
4. *For every three distinct indices, each of the four triples*
$$(p_{ij}, p_{ik}, p_{jk}), \quad (p_{ij}, q_{ik}, q_{jk}), \quad (q_{ij}, p_{ik}, q_{jk}), \quad (q_{ij}, q_{ik}, p_{jk})$$
has its minimum attained at least twice.
5. *There is no triple of indices with all $a_i, a_j, a_k > 4$ and $p_{ij} = p_{ik} = p_{jk} = 2$.*

The test uses $O(r^3)$ integer comparisons. Every rejection has a certificate involving at most three coordinate blocks.

Proof of Theorem 3.3. For paired scalar labels put $s_i = 2b_i$ and $w_{ij}^- = p_{ij} - 1$, $w_{ij}^+ = q_{ij} - 1$. The two contact triangles involving both labels of one block and a label of the other say

$$m_{ij} := \min(w_{ij}^-, w_{ij}^+) \le s_i, s_j, \qquad w_{ij}^- \ne w_{ij}^+ \Longrightarrow s_i = s_j = m_{ij}.$$

The even-character condition adds m_{ij} even $\Rightarrow m_{ij} = \min(s_i, s_j)$. These statements are exactly condition (2) after adding one. The four triangles involving three different blocks are precisely (4). Together they are equivalent to the unanchored paired-contact rank test. Here is the precise equivariant construction. In the paired contact array, let ι exchange paired labels and

put $s_x = h_{x,\iota x}$, with $s_x = \infty$ at a fixed origin. The conditions say that the array is ultrametric, every finite s_x is even, and

$$h_{xy} \text{ even} \quad \Longrightarrow \quad h_{xy} \geq \min(s_x, s_y).$$

At an attained level k, parents are the classes for $h_{xy} \geq k$ and children those for $h_{xy} > k$. In a parent preserved by ι, an odd k forces every child to be preserved, since an exchanged pair would have odd self-contact k. At even k at most one child is preserved: two such children would have mutual contact k and both self-contacts greater than k, contradicting the inequality. At odd degrees choose distinct coefficients on the preserved children. At even degrees give the preserved child coefficient zero and assign distinct opposite coefficients to exchanged pairs. Exchanged parents are treated in matching pairs; a distinguished origin always receives zero. These choices produce $F_{\iota x}(s) = -F_x(-s)$ with exactly the prescribed contacts. This constructs all the prescribed orders.

In an attached class put $t_i = a_i - 3$. The self/anchor triangle and its even-character condition say $t_i \leq s_i$ and t_i even $\Rightarrow t_i = s_i$. This is exactly (1). The remaining anchor triangles are (w_{ij}^-, t_i, t_j) and (w_{ij}^+, t_i, t_j); their minimum-twice condition is exactly (3) for attached indices. For an unattached index $a_i = 4$, condition (1) imposes no additional restriction beyond $b_i \geq 1$.

Condition (2) implies $p_{ij} > 2 \Leftrightarrow q_{ij} > 2$; condition (4) on the all-p triangle makes this relation transitive. Thus the contact classes are canonical. Condition (3) makes each class wholly attached or unattached. Condition (5) is precisely the at-most-two-attached-classes condition, since three distinct classes have all pair values equal to two. Theorem 3.1 therefore yields both necessity and sufficiency. □

Corollary 3.4. *A full-BC_r tuple is realisable if and only if all restrictions to at most three coordinate blocks are realisable in their full-BC supports. Two blocks do not suffice: on three blocks take all $b_i = 1$, all $a_i = 5$, and every $p_{ij} = q_{ij} = 2$. Every two-block restriction is realisable. The three attached classes exceed the capacity of the full tuple.*

3.5 Mixed blocks and the terminal capacity

In a recovered canonical classical frame, let S be the singleton indices and M the quadratic indices; a full A support uses its ordinary root-contact criterion, and low-rank identifications follow Section 3.2. For all remaining classical supports, the boundary factorisation extends as follows.

Proposition 3.5. *A positive marked tuple on a canonical classical frame outside full type A is realisable if and only if all of the following conditions hold.*

1. *Mixed singleton–quadratic entries are forced to be $p_{ij} = q_{ij} = 1$.*
2. *Without a terminal flag, test the singleton $D_{|S|}$ linear configuration $u_i \pm u_j$ by ordinary rational coefficient avoidance, using depths p, q. Empty configurations impose no test. Test the quadratic sector by the $C_{|M|}$ two-character criterion with positive depths*

$$d_{u_i - u_j} = p_{ij} - 1, \qquad d_{u_i + u_j} = q_{ij} - 1, \qquad d_{2u_i} = 2b_i.$$

   Its odd residues are $e_i + e_j$ on pair labels and zero on doubled labels. Apply this test to all quadratic indices together.
3. *With a terminal flag, test the singleton sector by the folded $B_{|S|}$ criterion, with depths $a_i - 1$ on u_i and $2p, 2q$ on $u_i \pm u_j$. For one singleton this is simply $a_i \geq 2$. The odd residue representation has all short roots equal to one nonzero vector and all long roots zero.*

4. *With a terminal flag, require in the quadratic sector $a_i \geq 4$, $p_{ij}, q_{ij} \geq 2$, and $p_{ij} > 2 \iff q_{ij} > 2$. Require $i \sim j \iff i = j$ or $p_{ij} > 2$ to be an equivalence relation, and decompose this sector into its contact classes. Each class has all $a_i = 4$ or all $a_i > 4$, and receives the same C or one-anchor BC rank test. The permitted number of attached classes is*

$$\kappa = \begin{cases} 2, & S = \varnothing \text{ or all singleton } a_i = 2, \\ 1, & \text{some singleton } a_i > 2. \end{cases}$$

All quantities in these tests are deterministic functions of the marked support and the proposed multiplicities.

Proof. A singleton has even s-order at least two, so every interaction with a quadratic block is transverse and contributes one. Without a terminal, the singleton and quadratic coefficients are independent. The rational singleton criterion is ordinary coefficient avoidance on the displayed linear forms; the quadratic criterion was proved above.

With a terminal, choose an active coordinate Y and call the other Z. The singleton entries are exactly those of its folded B model. If some singleton has $a_i > 2$, that model forces all singleton entries at least three and forces Z to have odd order at least three, or to be zero. Its interaction with every quadratic block contributes two, so only Y can attach a quadratic class: the budget is one. If all singleton entries are two, choose both terminals active with distinct leading squares; the budget is two. When there are no singletons an inactive terminal can likewise be made generically active without changing any quadratic contribution. This proves necessity of the budget.

For sufficiency, choose the singleton realisation first. In the one-anchor mode use its inactive Z, take $Y = s$, and apply the independent class constructions with at most one leading coefficient matching Y. In the two-anchor mode use $Y = s, Z = 2s$ and assign the at most two attached classes accordingly. The singleton–quadratic interactions remain forced to be one, all internal orders are correct, and the independent coefficient constructions give exactly the input spectrum. The same geometric realisation interface completes the proof. □

The role of the factorisation. The entries equal to two determine the partition into contact classes. After the independent class tests, the only remaining choice is whether the active terminal coordinates can accommodate those classes. The integer test reads the capacity directly from the singleton entries.

4 Exceptional supports and proof of the realisation theorem

Section 3 resolves every classical support. What additional supports and numerical images can arise from an exceptional Du Val section of type E_6, E_7, or E_8? Since the ambient rank is at most eight, curve rank $r \geq 5$ forces the internal subsystem to have rank at most three, yielding a clean ten-family classification (Theorem 4.1). We then tally all 378 exceptional marked types (Section 4.2), illustrate the low-rank contact plane on row 2e (Section 4.3.1), and assemble the full proof of Theorem 2.1 (Section 4.4); the individual exceptional root-lattice models are proved in Section 6.

4.1 The classification from rank five onwards

Apply (2.4) in the lattice of each folded core: for B_q the short depth is $n_s - 1$, the long depth is $2n_\ell$, and the odd-degree test uses $Q(B_q)/Q(D_q)$. The individual attachment proofs are carried out in Sections 6.1 and 6.2.

Theorem 4.1 (Complete intrinsic realisation for all $r \geq 5$)**.** *Let (Λ, B, n) be any finite candidate complete marked genus-zero GV spectrum of curve rank $r \geq 5$, with positive integer values on B and zero values elsewhere. Then it is realised by a projective small crepant contraction of complex local germs, with smooth source and isolated cDV target, if and only if one of the following input-intrinsic alternatives holds.*

1. *The support is classical and its all-positive classical criterion passes: the ordinary contact test in full type A, or the canonical classical factorisation with its singleton, quadratic, and terminal conditions.*
2. *The support is a full exceptional root system E_6, E_7, or E_8, and its ordinary rational contact test passes.*
3. *The support is one of the ten configurations in the following tables, and its displayed numerical criterion passes.*

Every accepted tuple has a polynomial Cartan witness and a geometric realisation.

Curve rank five. *Here Ω_k denotes the fundamental-weight orbit of A_5.*

intrinsic signed support type	$\lvert B\rvert$	*numerical criterion*
$A_5 \cup \Omega_3$	25	Ω_3 *values* 1; A_5 *ordinary contacts*
$A_5 \cup \Omega_2 \cup \Omega_4$	30	*weight values* 1; A_5 *ordinary contacts*
$(B_4 + A_1)$ *spin attachment*	33	*extra weights* 1; B_4 *folded*; A_1 *arbitrary*
B_5 *spin extension*	41	*spin values* 1; B_5 *folded*
$A_5 \cup \bigcup_{k=1}^{5} \Omega_k$	46	$\Omega_{1,5} : 2$; $\Omega_{2,3,4} : 1$; A_5 *ordinary*
F_4 *line attachment*	50	*Theorem 6.7.*

The lattices in these rows are those in their intrinsic definitions: respectively $Q(A_5) + \mathbb{Z}\omega_3$, $Q(A_5) + \mathbb{Z}\omega_2$, the index-two $B_4 + A_1$ attachment lattice, the spin-extended B_5 lattice, $P(A_5)$, and $Q(F_4) \oplus \mathbb{Z}v$. In the 50-label row, the criterion requires all mixed labels $s \pm v$ to have value one, the full F_4 test on the core, the length-two radial test on $(a, b) = (n_v, n_{2v})$, and the single compatibility condition

$$a = 4 \quad \text{or} \quad \min_{s \text{ short in } F_4} n_s = 2.$$

Curve ranks six and seven.

r	*intrinsic signed support type*	$\lvert B\rvert$	*numerical criterion*
6	D_6 *plus one half-spin orbit*	46	*spin values* 1; D_6 *ordinary contacts*
6	E_6 *plus* $27 \cup 27^*$	63	*weight values* 1; E_6 *ordinary contacts*
6	B_6 *spin extension*	68	*spin values* 1; B_6 *folded*
7	E_7 *plus its minuscule orbit*	91	*weight values* 1; E_7 *ordinary contacts.*

Their marking lattices are, respectively, the half-spin extension of $Q(D_6)$, $P(E_6)$, $P(D_6)$, and $P(E_7)$. There are no additional nonclassical attachment rows in curve rank eight or above.

Proof of Theorem 4.1. The geometric interface gives every realisation an ADE sectional presentation. If the ambient root system is classical, the proved infinite-family reduction and its all-positive numerical criterion give alternative (1). This accounts for classical presentations of arbitrary ambient rank.

An exceptional ambient root system has rank at most eight. Hence, for $r \geq 5$, the internal subsystem has rank at most three. The possible internal types are

$$\varnothing, \quad A_1, \quad A_2, \quad A_1^2, \quad A_3, \quad A_2 + A_1, \quad A_1^3.$$

No internal roots gives alternative (2). For curve rank five, the remaining possibilities are precisely

$$E_6/A_1; \qquad E_7/A_2,\ E_7/A_1^2; \qquad E_8/A_3,\ E_8/(A_2+A_1),\ E_8/A_1^3.$$

Their intrinsic supports and complete numerical images are exactly the six rank-five rows: the ordinary minuscule attachments, the $B_4 + A_1$ spin attachment, the spin extension of B_5, the index-six ordinary attachment, and the F_4 line attachment. Sections 6.1 and 6.2 give the constructions and compare the markings for these six rows.

In curve rank six, the nonempty internal cases are E_7/A_1, E_8/A_2, and E_8/A_1^2, giving the three displayed rank-six rows. In curve rank seven only E_8/A_1 remains, giving the last row. In curve rank eight an exceptional presentation is the full E_8 system with $I = \varnothing$; in ranks at least nine there is no exceptional ambient possibility.

Conversely, the all-positive classical and full-root tests have their established smooth-source constructions. Each attachment criterion has been proved necessary and sufficient with a polynomial Cartan construction, including every boundary value one or two. Thus every accepted tuple is realised with precisely its prescribed complete support and numbers. This proves the equivalence. □

4.2 The exceptional support inventory

We count the support types in the first alternative of Theorem 2.1. Two positive supports are identified only when a permutation of their chamber-generator bases identifies their complete label sets. Different positive chamber types are therefore counted separately unless such a based integral identification exists.

There are 445 exceptional presentations with $J \neq \varnothing$ in E_6, E_7, E_8. Their positive projected supports give 378 marked types. Including full reduced supports, full BC, every classical enhanced support, the spin extensions of B_3, B_5, B_6, the five- and eight-label rank-two attachments, the eight ordinary attachment families, the diagonal attachment to B_3, the $E_7/(A_1+A_1)$ family, the F_4–line criterion, the doubled-short G_2 criterion, the three nine-label chamber types, and the separately cited rank-one classification, the union of these families contains 167 of the marked types. The counts in each rank are given in Appendix B.2. The union count removes overlaps: for example the five-label partial doublings are already included among the classical enhanced supports. The remaining types are pairwise partitioned by the signed configurations in Tables A.1–A.3.

The nine-label configuration of Section 6.3.5 contributes three additional marked chamber types, represented by $J = 16, 17, 57$ in zero-based E_8 numbering (one-based $27, 28, 68$). Its complete criterion accounts for all entries, including the three forced units and the independent radial choices. The 27 additional rows in Appendix A account for 25 further marked types in rank two, 83 in rank three, and 103 in rank four. The four rank-one supports of lengths three through six enter through the single-curve classification of Section 2.1.1.

For each of the 445 presentations, compare its full signed set first with the families just listed and then with the vector columns in Appendix A. Proposition 2.4 tests the entire set and the determinant in the quotient lattices. The first comparison constructs the ordinary weight orbits and the folded and rank-two models of Section 6, as well as the classical frames with every choice of doubled axes. Every positive type belongs to the first union or to a unique additional row. Appendix B.2 gives the rank counts and chamber maps. Across ambient types the numerical images are compared separately in Section 4.4.

The following count applies to the index-two and index-three families.

Lemma 4.2. *Let I be a parabolic subsystem of a simply laced root system Φ, and let $c(I)$ be*

its number of connected components. Then

$$2|B| = \frac{1}{|W_I|} \sum_{w \in W_I} |\Phi^w| - c(I). \tag{4.1}$$

Proof. Each nonzero restricted-root fibre is a single W_I orbit. One way to verify this familiar fact is to project an external root to the internal Cartan. Its pairings with internal roots lie in $\{0, 1, -1\}$, so its dominant internal weight is zero or minuscule on each component. Two roots in one restricted fibre differ by $Q(I)$; distinct dominant minuscule weights represent distinct classes in $P(I)/Q(I)$. Hence they lie in the same internal Weyl orbit. There are exactly $c(I)$ internal root orbits. Burnside's formula counts all orbits, and the external ones occur in opposite pairs. □

4.3 Contact orders on the low-rank exceptional supports

For each of the 27 signed supports in Appendix A, the corresponding row specifies which GV multiplicities are fixed and which can vary together. To apply the row, first identify its support with the input curve classes.

For a row S try integral isomorphisms $\phi : \mathbb{Z}^r \to \Lambda$ such that $\phi(S) = B \cup (-B)$. Read each coefficient at the positive representative of $\phi(t)$. Use the pullback of the *input* positive chamber on S. Its primitive inward wall normals (the atoms of the positive-label monoid) must map to the c_i in their given order. Thus an input may pull back to a different chamber from the chamber used in the table. The table vectors include the standard basis; the images of these vectors range over a finite set, and one checks determinant ± 1, the complete image and the chamber marking.

Once the marking is fixed, compare the prescribed multiplicities with the fixed entries of the row. The conditions following the table test the remaining entries. Section 4.3.1 works this out for a ten-label support.

4.3.1 Worked example: two orders on a ten-label support (row 2e)

In row 2e of Table A.1, put $w = (0, 1)$ and $v = (1, 2)$. The four labels $v \pm 2w$ and $2v \pm w$ have multiplicity one. The six other labels are $w, 2w, v - w, v, v + w, 2v$. Put $A = n(2w)$, $D' = n(2v)$, $R = n(v - w)$ and $S = n(v + w)$. Fix $n(w) = n(v) = 6$. In the mixed case $R, S \geq 5$ the cover has $t = s^4$. Four roots over each of $2w, 2v$ already have order one, leaving two roots in each fibre. Twelve roots over each of $v - w, v + w$ have order one, leaving four in each fibre. The roots remaining in a given fibre have the same order. Thus, for example, $4R = 12 + 4u$, where u is the required order over $v - w$. Along the four labels $2w, 2v, v - w, v + w$ the orders to prescribe are

$$2n(2w) - 2, \qquad 2n(2v) - 2, \qquad R - 3, \qquad S - 3.$$

The fibre counts and the four restrictions below are the degree-one and degree-two root calculations for E_8 with internal system $A_3 + A_3$; see Appendix B.1 for the explicit coordinates.

Take $n(2w) = n(2v) = 2$. If $R = 6$ and $S = 5$, the four required orders are $(2, 2, 3, 2)$. In coordinates (x, y) on the degree-two coefficient space the corresponding linear forms are

$$-2x, \qquad -2y, \qquad -x - y, \qquad x - y.$$

At $(x, y) = (1, -1)$ the four values are $(-2, 2, 0, 2)$, so precisely the third form vanishes. Both mixed labels have the same nonzero form at degree three, so we can give the surviving one order exactly three. The fourth label already has its order-two term. Exchanging R and S uses $(x, y) = (1, 1)$ at degree two.

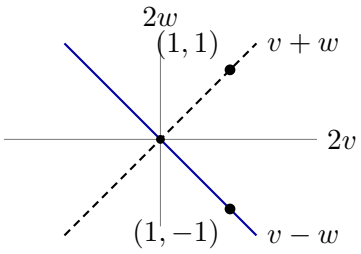


Figure 1: The mixed branch of row 2e. Each line is the kernel for the indicated label in the real slice of the degree-two coefficient space. The two marked points each lie on exactly one mixed-label kernel.

Now ask for $R = S = 6$. Both mixed forms must vanish at degree two. Their two lines meet only at zero. At zero the two doubled forms also vanish, although both must have order two. This excludes $(2, 6, 6, 2)$ in the order (A, R, S, D'). If $A = D' = 3$, the required orders become $(4, 4, 3, 3)$. We can now set the entire degree-two coefficient to zero, choose a degree-three coefficient nonzero on the common mixed form, and choose a degree-four coefficient nonzero on both doubled forms. This gives $(3, 6, 6, 3)$. The change in the doubled values changes precisely the condition that forced the contradiction.

4.4 Proof of the realisation theorem

For the three divisors of Section 2.2.1, one presentation forced $n_{10} = n_{01} = 1$, although the same marked support also admitted $(2, 2, 3)$. For the general proof we must therefore answer two questions. Which presentations have the prescribed signed support in its curve lattice? For these presentations, what is the union of their numerical images? The first question involves only roots and lattices. The second concerns all positive integer values at once.

Smoothness enters the second question before any orders can be chosen freely. For example, the first coefficient in row 3i has to be nonzero on an internal root as well as satisfy the seven displayed order conditions. This gives $x_7 > 2 \Rightarrow x_2 = x_3 = 4$. In row 4f the same two primitive cubic coefficients occur in both G_2 cores. Their ratio can be a cube root of unity or the negative of one. Requiring the two distinct matches simultaneously would force a nonzero ratio to lie in two disjoint sets.

For a signed support S in Appendix A, write $P(n)$ for its fixed entries and its conditions on the remaining columns. To compare this rule with geometry, identify a compatible presentation p with S by an integral map T as in Proposition 2.4. Write $Q_{p,T}(n)$ for the condition of Section 2.3.3 in those coordinates. Corollary 2.6 shows that the two implications needed for this row are

$$Q_{p,T}(n) \implies P(n) \quad \text{for every } (p, T), \qquad P(n) \implies \bigvee_{p,T} Q_{p,T}(n). \tag{4.2}$$

The first is necessity for each possible section. The second is sufficiency: for the given input it supplies a presentation and a single Cartan arc with all the required multiplicities.

Proposition 4.3. *Each of the* 27 *numerical rows in Appendix A equals the union of the smooth Cartan images in the ordered marking. Of the* 228 *compatible exceptional presentations,* 226 *give the entire corresponding row individually. The two proper inclusions occur in row 3f.*

Computer-assisted proof. The support calculation of Appendix B.2 exhausts the exceptional presentations and excludes classical competitors for these rows. Two identifications of a signed support with the same target differ by an automorphism of the signed set. The integral basis

test of Proposition 2.4 enumerates their permutations on sign-pairs. For each generator g, we test whether $P(n)$ and $P(gn)$ differ on any positive input. They agree for all 66 generators of the groups, which have 734 permutations in total. Thus one identification per presentation suffices for the forward implication. The reverse implication is tested on the union of the chosen images. In each based type there is also an individual presentation giving the entire row.

On the geometric side, (2.6) gives the exact conditions on each initial pattern. On the table side, Appendix A specifies the constants, inequalities and span tests. For a span test one may use the maximal proper flats: if a target lies outside the span of the deeper forms, extend a basis of that span together with the target to a basis chosen from the forms. Removing the target gives a separating hyperplane. This works over the rational and residue fields used in the tables.

For every admissible initial pattern the conjunction of its conditions with $\neg P(n)$ has no positive integer solution. Conversely $P(n) \wedge \neg \bigvee_p Q_{p,T}(n)$ has no solution for any of the 27 rows. For 226 individual presentations the same reverse test already holds with that p alone. For the two E_8 presentations of row 3f, the tuple in (4.3) belongs to the row and is excluded from both images. Appendix B.3.1 gives the counts and Appendix B.3.4 gives the commands for both tests. These calculations establish (4.2) over all positive integers. □

Proof of Theorem 2.1. A realising contraction has a Du Val section. The nonzero images of its positive roots give precisely its marked support. For a classical presentation Section 3 gives the numerical conditions. For an exceptional presentation, Section 4.2 places its support in the first alternative of the theorem or in one of the 27 rows. The family calculations in Section 6 and the full-root tests in Section 2.2.6 give necessity in the first alternative, with classical overlaps as in Appendix B.3.3. In the second alternative necessity follows from Proposition 4.3.

Conversely, a passing classical or individual family test gives its polynomial construction in Sections 3 and 6, or by the full-root construction in Section 2.2.6. For a passing row, the reverse implication in (4.2) supplies a true $Q_{p,T}(n)$. Section 2.3.3 constructs one polynomial with those fibre sums and the required smooth internal orders. Its pullback gives a local algebraic contraction by Section 2.2.2. The support identification accounts for every nonzero label and transports the ordered marking.

All tests use finite integral support comparisons and exact linear algebra. For each additional row the rank and the configurations of forms are fixed, so the stated cost bound follows from arithmetic on the binary input. □

4.5 A strict inclusion on sixteen labels (row 3f)

In a lattice with basis u_1, u_2, u_3, row 3f has the sign-pairs of $u_i, 2u_i, u_i \pm u_j$ together with the four pairs $\pm u_1 \pm u_2 \pm u_3$. Consider the tuple

$$n_{u_i} = 5, \qquad n_{2u_i} = 1, \qquad n_{u_i \pm u_j} = 3, \qquad n_{\pm u_1 \pm u_2 \pm u_3} = 1, \tag{4.3}$$

with all four fixed entries equal to 1. In the variable-column order of Table A.2 this is

$$(3, 5, 5, 1, 3, 1, 3, 3, 5, 3, 3, 1). \tag{4.4}$$

Its three axes form a single active class in which every depth equals 2. We determine which presentation realises it.

Work in the Euclidean E_8 root system $\pm e_i \pm e_j$ and $\frac{1}{2}\sum \epsilon_i e_i$ ($\prod \epsilon_i = 1$). Setting $v = u_1$, $w = u_2 - u_3$, and $t_0 = u_2 + u_3$, consider the quotient map sending $e_2, e_3 \mapsto v$, $e_1 \mapsto w$, $e_8 \mapsto t_0$, and $e_4, \ldots, e_7 \mapsto 0$. The integer roots project up to sign to $v, w, t_0, 2v, t_0 \pm w, v \pm w, v \pm t_0$, while the half-roots project to $\pm u_2, \pm u_3$ with an optional summand $\pm v$, recovering the complete

support and lattice of row 3f. The kernel roots form D_4 on $e_4, \dots, e_7$ together with A_1 on $e_2 - e_3$. Restricting to the E_7 subsystem $x_4 = x_5$ reduces the kernel roots to the four orthogonal pairs generated by

$$e_2 - e_3, \quad e_4 + e_5, \quad e_6 - e_7, \quad e_6 + e_7.$$

In either ambient system, a generic functional of the three quotient coordinates vanishes precisely on these kernel roots, so both presentations are parabolic. The integral kernel in E_8 is given by $x_1 = x_8 = 0$, $x_2 + x_3 = 0$ and $x_4 + x_5 + x_6 + x_7$ even. This is the root lattice of $D_4 + A_1$. Its intersection with $x_4 = x_5$ is generated by the four displayed roots.

An E_7 polynomial. On the double cover $t = z^2$, put $H_i = h_i z^2$ with $(h_1, h_2, h_3) = (1, 2, 4)$, $A = 2z$, and $B_i = z$, and take the eight-coordinate arc

$$(H_2 - H_3,\ H_1 + A,\ H_1 - A,\ B_1,\ B_1,\ B_2 + B_3,\ B_2 - B_3,\ H_2 + H_3).$$

The product of the four internal reflections exchanges the second and third coordinates and negates the moving coordinates on $x_4 = x_5$, and all four internal forms have order 1. For distinct i, j, k, evaluation on the integer and half-integer roots with $x_4 = x_5$ gives the following orders.

label	forms in its fibre (independent moving signs)	orders in z
$2u_i$	$2H_i$	2
$u_i \pm u_j$	$H_i \pm H_j \ \pm B_i \pm B_j$	$1, 1, 2, 2$
u_i	$H_i \pm A \pm B_j \pm B_k$	$1, 1, 1, 1, 1, 1, 2, 2$
$\pm u_1 \pm u_2 \pm u_3$	$\pm H_1 \pm H_2 \pm H_3 \ \pm A$	$1, 1$

For instance, over u_1 the eight roots $e_l \pm e_a$ ($l \in \{2, 3\}$, $a \in \{6, 7\}$) have exactly two vanishing linear terms, while in the half-roots the equality of signs $4, 5$ fixes the parity of signs $6, 7$ from the remaining four signs, yielding the forms over u_2, u_3. Together with the 4 positive internal roots, these $3 + 24 + 24 + 8 = 59$ external roots account for all 63 positive roots of E_7, and summing each row and dividing by 2 recovers (4.3). Since the same fibre forms hold for arbitrary centred even H_i and odd A, B_i of order 1, permuting (H_i, B_i) and negating H_i shows that the E_7 image is invariant under all signed permutations of the doubled-ray basis.

The obstruction in E_8. Choose a D_4 Coxeter element acting as minus the cycle $\sigma = (4\,5\,6)$ and negating e_7 (the standard negative three-cycle model up to two sign changes), while the A_1 reflection exchanges e_2, e_3. On the sixfold cover $t = s^6$, the degree-one primitive eigenvector on positions $4, 5, 6$ has entries proportional in some order to $(1, \omega, \omega^2)$ ($\omega^3 = 1$, $\omega \neq 1$), with nonzero coefficient by smoothness. The degree-two eigenspace is zero. At degree three, the coefficients on e_2, e_3 are $Y, -Y$ with $Y \neq 0$ by A_1 smoothness, the coefficient on e_7 is some X, and the coefficient on e_1 vanishes.

Over v, twelve of the sixteen roots $e_l \pm e_a$ ($l \in \{2, 3\}$, $4 \leq a \leq 7$) exit at degree 1, while the remaining four evaluate to $\pm Y \pm X$ at degree 3. Over w, six of the eight roots $e_1 \pm e_a$ exit at degree 1, and the remaining two evaluate to $\pm X$ at degree 3. If all surviving roots exited at degree 3, the invariants would be $(12 + 4 \cdot 3)/6 = 4$ and $(6 + 2 \cdot 3)/6 = 2$. Reaching $n_w = 3$ forces $X = 0$, whereas reaching $n_v = 5$ forces one of $\pm Y \pm X$ to vanish, contradicting $Y \neq 0$.

Containment for every order. We also construct a direct map from the E_8 image into the E_7 image. Set $W = \{x_4 = x_5 = x_6\}$ and let R be the set of roots in W. By $1 + \omega + \omega^2 = 0$, a rational root vanishes on the degree-one eigenvector precisely when its entries in positions $4, 5, 6$ agree; all E_8 roots outside R therefore exit at degree 1. The cycle σ preserves labels. A half-root outside R has two equal entries among positions $4, 5, 6$, while an integer root with

three distinct entries there has zero projection; hence every external root outside R has a σ-orbit of size 3 meeting E_7 in a unique root.

Averaging an E_8 arc over $1, \sigma, \sigma^2$ projects it onto W, where the Coxeter element acts as the E_7 involution, producing an arc $\eta(z)$ in $z = s^3$ that preserves every evaluation on R (including $e_2 - e_3$). The vector $\xi = e_4 + e_5 - 2e_6$ lies in the negative E_7 eigenspace, annihilates R, and is nonzero on every E_7 root outside R. For all but finitely many scalars M, the arc $\eta(z) + Mz\xi$ gives order 1 to all E_7 roots outside R while keeping all orders on R unchanged. In each fibre, three order-one roots on the sixfold cover contribute as much as one on the double cover. This proves containment for all orders. The tuple in (4.3) makes it strict. In the numbered diagrams, opposition on edge 12 interchanges the two $4A_1$ complements $135, 235$ in E_7, and opposition on edge 67 interchanges the two $D_4 + A_1$ complements $156, 157$ in E_8; signed-axis invariance transports (4.3) to both markings. Since classical rank-three frames have at most 12 positive labels and the census of Section 4.2 exhausts all 111 exceptional triples J, no other presentation can intervene.

5 Restrictions on the multiplicities and counts of spectra

Write $U(n)$ for the number of entries equal to one. We first ask what $U(n)$ tells us about the possible support. We then fix the support and count how many spectra fit in a box of side N.

5.1 The sharp unit-entry bound

For a positive marked tuple on B let

$$U(n) = \#\{\beta \in B : n_\beta = 1\}.$$

Call a support *full* if it is the full positive support of an irreducible reduced crystallographic root system or of BC_r, with its based integral marking. This definition is a finite intrinsic condition on the input support.

Theorem 5.1. *Let n be a complete GV spectrum of a smooth threefold contraction over one isolated cDV point, of curve rank $r \geq 2$. If its support is neither full nor a spin extension of B_3, then*

$$U(n) \geq 2r - 2.$$

For every $r \geq 2$ there is a nonfull support, distinct from the spin extension, and a smooth-source spectrum on it with exactly $2r - 2$ unit entries. Every other entry of this example is either two or four.

The exception is specific: the spin extension exists only in rank three, its four spin entries always equal two, and its number of unit entries is either zero, three, four, five or six. Counting the unit entries can therefore exclude a proposed spectrum before choosing a presentation.

5.1.1 Forced unit fibres

The number of forced units can be read from the Weyl orbits in individual root fibres. For a choice of J in E_6, E_7, E_8, put $I = \Delta \setminus J$ and $R_\beta = \{\alpha \in \Phi^+ : \pi_I \alpha = \beta\}$.

Lemma 5.2. *Suppose an internal component K is of type A_k, a nonzero fibre R_β is one W_K orbit of size $k + 1$, its projected weight has squared norm $k/(k + 1)$ with roots of squared norm two, and every other internal component acts trivially on that fibre. Then every smooth admissible Cartan arc has $n_\beta = 1$.*

Proof. These weights form the standard or dual-standard A_k orbit. In standard diagonal coordinates a primitive Coxeter eigenvector has distinct entries proportional to $(1, \zeta_{k+1}, \dots, \zeta_{k+1}^k)$, with none zero. Every weight in this orbit therefore has nonzero evaluation on the mandatory primitive coefficient. If the common Cartan cover has degree m, each root in this fibre has order $m/(k+1)$. Other internal components are orthogonal to these root forms, so cannot change this leading order, and invariant coefficients begin at degree m. Summing the $k+1$ equal orders and dividing by m gives $n_\beta = 1$. □

Lemma 5.3. *There are 445 choices of a nonempty subset J of the simple vertices in E_6, E_7, E_8. Exactly 29 have full support, including 18 of curve rank at least two. The four spin-extension presentations have no unit fibres of Lemma 5.2. For every other nonfull presentation the number of presentations and the minimum number of fibres satisfying Lemma 5.2 are given by rank in Appendix B.3.3. Among the 52 rank-two rows, exactly twelve have only two such fibres: six have the five-label partial doubling of B_2, and six have the eight-label attachment of Section 6.3.5. The other forty have at least three certified unit fibres.*

The count follows by enumerating the projected roots and testing the unit-fibre condition of Lemma 5.2; Appendix B.3.3 gives the orbit test, the complete histogram, and its verification.

5.1.2 Proof of the unit bound

For an exceptional presentation, Lemma 5.3 gives the claimed bound in every rank; no exceptional presentation has curve rank above eight, and the full rank-eight presentation is full. For type A the support is full. Fix the type-D presentation of the given contraction and let its p singleton blocks and q nonsingleton blocks satisfy $p + q = r$. If $q = 0$, the frame gives D_r without a terminal component and B_r with one. If $p = 0$, the two supports are C_r and BC_r, respectively. Here $D_3 = A_3$ and the disconnected D_2 is excluded. Thus on a nonfull support this presentation has $p, q \geq 1$. For a singleton and a block of size $d \geq 2$, either signed root fibre is a standard or dual-standard orbit of the internal A_{d-1}, of size d and weight norm $(d-1)/d$. All other internal components act trivially on this fibre. Lemma 5.2 therefore gives multiplicity one for each sign. They are distinct labels, and pairs give distinct labels. Consequently

$$U(n) \geq 2pq \geq 2(r-1).$$

This proves the necessary bound.

The construction in Section 5.1.3 attains the bound in every rank.

5.1.3 A bound attained in every rank

For equality in Theorem 5.1, use D_{r+3} with $r-1$ singleton blocks, one quadratic block and terminal D_2. On $t = s^2$ take coordinates

$$x_i = c_i s^4 \quad (1 \leq i \leq r-1), \qquad x_r^+ = 3s + s^4, \quad x_r^- = -3s + s^4, \qquad Y = s, \quad Z = 2s,$$

where the c_i are distinct positive constants. Internal roots $x_r^+ - x_r^-$ and $Y \pm Z$ have order one. The complete positive support is the terminal classical support with exactly its last primitive axis doubled. Its multiplicities are

labels	n
$u_i \pm u_r$ $(i < r)$	1
$u_i \pm u_j$ $(i < j < r)$	2
u_i $(i < r)$	2
$2u_r$	2
u_r	4.

Every root evaluation is nonzero. The Cartan–GV interface gives a smooth-source realisation, and exactly $2r-2$ distinct entries equal one. This gives the equality case of Theorem 5.1.

5.1.4 Seven values without a curve marking

Can $(6,5,3,2,2,2,1)$ occur as the complete list of nonzero multiplicities for some assignment of curve classes? A single curve gives at most six terms. For exceptional presentations the only seven-label family is row 2b of Table A.1, whose multiplicities, up to permutation, are $(5,3,3,1,1,1,k)$ with $k \geq 1$. Among classical supports, seven labels occur only in the rank-three nonterminal frame with one quadratic block and two singletons. Each singleton gives two mixed labels of multiplicity one (Section 3.5). Thus every seven-term spectrum has at least three entries equal to one. The tuple $(6,5,3,2,2,2,1)$ has only one, so no assignment of its entries to curve classes can realise it.

5.2 Supports with few unit entries

5.2.1 The spin extension of B_3

Use the three sets of labels from Section 6.2.1. The set S consists of the short roots, L of the long roots, and C of the four spin labels.

Corollary 5.4. *Require $n_c = 2$ for $c \in C$ and $n_s = 2$ for $s \in S$. Then the remaining six values on L are realisable if and only if they are an ordinary positive $D_3 \simeq A_3$ root-contact function. Equivalently their strict superlevels are rational root flats in the long-root system. Thus every full A_3 spectrum extends by seven constant entries equal to two to a spectrum on this 13-label noncrystallographic support. In this extension identify $Q(A_3)$ with the long-root lattice $Q_L = \mathbb{Z}L$, with its induced positive chamber, and then use $Q_L \subset \Lambda$. This inclusion has index $[\Lambda : Q_L] = 4$; the resulting curve marking is the chamber basis in Λ. In particular, all thirteen values can equal two on either marked chamber of the spin extension.*

Proof. The three short normalised depths are one, so the moving degree one removes all short forms and imposes no condition on the long depths, which are positive even integers. The remaining even-level span tests are exactly the ordinary rational root-contact tests for the six long roots. Their root system is $D_3 = A_3$. The constant long-root tuple of value two passes these tests in both chambers. Section 5.2.3 gives a polynomial witness at this sharp boundary value. □

The six long labels restrict to an ordinary A_3 contact function in every spin-extended spectrum, even when the three short entries are not equal. Indeed, at every even long depth, the rational test with all higher roots implies the same test with just the higher long roots. Thus the restricted strict superlevels are root flats of $D_3 = A_3$.

There is a useful intrinsic incidence description. Regard the four spin labels C as vertices of a complete graph. For each unordered pair $\{c, c'\}$, exactly one of $c + c'$ and $c - c'$, up to sign, is a long label. These six labels are distinct, so they identify L with the edges of K_4. After independently changing the signs of the four spin weights, they form the weights of one half-spin representation of D_3; their pairwise differences are the long roots. Consequently the six long multiplicities are precisely pairwise contact orders on four branches.

Proposition 5.5. *In a spin-extended B_3 spectrum, unit entries occur only on the long labels. If they occur, their edges form the complete multipartite graph associated to a nontrivial partition of the four spin vertices. In particular their number is one of $3, 4, 5, 6$; each of these four possibilities is realisable.*

Proof. Spin entries equal two and short entries are at least two. For the ordinary A_3 contact matrix on the long edges, the relation $i \sim j$ if the contact exceeds one is an equivalence relation. This follows either from valuation of differences or the three-root rational span test. The unit edges are exactly those joining distinct equivalence classes. The nontrivial block sizes are $(3,1)$, $(2,2)$, $(2,1,1)$ and $(1,1,1,1)$, with respectively $3,4,5,6$ crossing edges. All occur by assigning contact one between blocks and contact two within each block, then using the ordinary-A_3 slice with all seven other entries equal to two. □

5.2.2 The criterion by number of unit entries

Theorem 5.6. *Let n be a complete positive marked tuple of curve rank $r \geq 2$, and write $U = U(n)$. Suppose $U < 2r-2$ or $U \leq 2$. Then n is realisable if and only if its marked support occurs in the applicable row and its multiplicities pass the test for that support:*

number of units	*possible supports*
$U = 0$ *or* $3 \leq U < 2r-2$	*full reduced,* BC_r, *spin extension of* B_3 $(r=3)$
$U = 1$	C_r, BC_r
$U = 2$, $r \geq 3$	C_r, BC_r
$U = 2$, $r = 2$	A_2, $B_2 = C_2$, G_2, BC_2, *five- or eight-label attachment*

For full reduced supports use the span test in (2.4); for full BC_r use Theorem 3.3. For the spin extension of B_3 use Theorem 6.7. For the five- and eight-label attachments use the corresponding rows of Theorem 6.13; the doubled coordinate P must be at least two to give exactly two units. Each acceptance gives a polynomial Cartan arc.

Proof. If $U < 2r-2$, Theorem 5.1 leaves only the full supports and the spin extension of B_3. The only additional case with $U \leq 2$ is $r = 2$, $U = 2$. The classical frames then give just one further support, the five-label partial doubling. For the nonfull exceptional presentations, Lemma 5.3 leaves the five- and eight-label attachments. Their criteria give $U = 2$ exactly when $P \geq 2$.

To distinguish one or two units on the full supports, first note that in a reduced nonsimply-laced system every short multiplicity is at least the fold order. A nonempty set of unit edges on an A_3 subsystem is the cut of a nontrivial partition of four vertices and has at least three positive labels. Every root of an irreducible ADE system of rank at least three lies in such a subsystem, as does every long root of B_r $(r \geq 3)$ and F_4. This leaves C_r, with long roots of type A_1^r, and full BC_r. The spin extension has at least three units whenever it has any, by Proposition 5.5. In rank two a unit on full A_2, or on the long A_2 of G_2, forces a second unit by the minimum condition. This gives the rows in the table. Conversely the indicated support test constructs an arc with precisely the proposed multiplicities. □

In rank one, a spectrum with $U = 0$ has support A_1 or BC_1: every row of length three through six in Section 2.1.1 contains a unit. The same list gives the allowed values on A_1 and BC_1.

5.2.3 Thirteen entries equal to two on the spin extension of B_3

The spin extension has three short, six long and four spin labels. What happens if each is assigned the value two?

Polynomial witness for the constant case of Corollary 5.4. The normalised short depths are one and the long depths are four. The odd test at one sees only long roots above it and passes. The even test at four has no higher roots and passes. An explicit E_8/D_5 witness is

$$\lambda(s) = s\xi + s^4(a-b) + s^{16}z,$$

where ξ is a regular primitive D_5 Coxeter eigenvector and a, b are the two exchanged outer simple roots of its surviving D_4 in Section 6.2.4, and z is the sum of its positive roots. The middle coefficient is anti-invariant, nonzero on every short orbit, and zero on long roots. The last coefficient is fixed and regular. The four spin fibres contribute $16/8 = 2$; each short fibre contributes $(8 + 2 \cdot 4)/8 = 2$; and each long fibre contributes $16/8 = 2$. □

5.3 Cofinality and repeated multiplicities

Corollary 5.7. *For a marked support B the following are equivalent:*

1. *for every N some realisation on B has all entries at least N;*
2. *some realisation on B has all entries at least three;*
3. *B is a full irreducible reduced crystallographic support or full BC_r.*

On the region $n_\beta \geq 3$, the reduced-root rational/residue test and the one-class BC test are intrinsic necessary and sufficient criteria. The threshold cannot be replaced by two.

Proof. For $r \geq 2$ the unit gap excludes every support except the full supports and spin-extended B_3 when all entries are at least three. The spin entries in that last support are always two, so it is excluded as well. The rank-one assertion follows from the complete list in Section 2.1.1. Conversely, full ADE admits the constant tuple N. In a nonsimply-laced reduced root system of fold order e, take long values N and short values $eN + e - 1$; all normalised depths then equal eN, so coefficient avoidance passes. For full BC_r use $n_{2u_i} = N$, $n_{u_i} = 2N + 3$ and $n_{u_i \pm u_j} = 2N + 1$. These give one attached class with every residual depth $2N$.

If every BC entry is at least three, all pair entries exceed two, so the canonical partition has one class. Its anchored rank test, also allowing the unattached value $a_i = 4$, is the global one-anchor test. This proves the claimed intrinsic criterion. The constant all-two spectrum on the nonfull spin extension of B_3 proves sharpness. □

Theorem 5.8. *Every complete marked GV spectrum of a smooth-source contraction to one isolated cDV point, with $r \geq 2$, has distinct labels β, γ such that $n_\beta = n_\gamma$. Equivalently, numbering the $M = |B|$ labels,*

$$\prod_{i<j} (n_i - n_j) = 0.$$

Proof. A nonfull support other than spin-extended B_3 has at least $2r - 2 \geq 2$ unit entries. The spin extension has four entries equal to two. It remains to consider full supports.

In ADE, a parabolic A_2 subsystem supplies three ordinary contact orders whose minimum occurs twice. The same argument applies to the long D_4 roots of F_4 and the long A_2 roots of G_2. For B_r, use its folded scalar model: long entries are the orders of $x_i - x_j$ and $x_i + x_j$, and a short entry is $1 + \min\{\operatorname{ord}_s x_i(s^2), \operatorname{ord}_s z(s)\}$ for one common odd z. If the two long entries for a pair agree, there is already a repeated value. Otherwise the two x's have the same order, and the two short entries agree. For C_r, the paired scalar model has self-contact $2n_{2u_i}$ and signed pair contacts $n_{u_i \pm u_j} - 1$. If the two latter contacts differ, the two contact triangles force both self-contacts to equal their minimum, so the doubled entries agree; if they coincide, the original pair entries agree.

For full BC_r, condition (2) of Theorem 3.3 says directly that either $p_{ij} = q_{ij}$ or $b_i = b_j$ for any pair. This finishes every case. □

The rank hypothesis is sharp. The length-two row of Section 2.1.1 realises $(4, 16)$ on one curve.

Corollary 5.9. *For the spin-B_5 and spin-B_6 supports, respectively,*

$$U(n) = 16 \quad \text{or} \quad U(n) \geq 24, \qquad U(n) = 32 \quad \text{or} \quad U(n) \geq 42.$$

For the $B_4 + A_1$ spin attachment,

$$U(n) \in \{16, 17\} \quad \text{or} \quad U(n) \geq 22.$$

The lower values and the first value above each gap are attained.

Proof. The short entries are at least two. The extra weights give sixteen, thirty-two, and sixteen units respectively; the independent A_1 entry in the third family can add zero or one. The long entries are ordinary D_q contacts. Write a witness as $x(t) = ta + O(t^2)$. If $a = 0$, no long entry is one. Otherwise let m be the number of nonzero coordinates of a. Mixed zero/nonzero pairs give $2m(q-m)$ unit labels, and each pair of nonzero coordinates gives at least one. The number of long units is consequently at least

$$2m(q-m) + \binom{m}{2} \;\geq\; 2(q-1), \qquad q \geq 4.$$

For the last inequality the expression is concave in m on $[1, q]$; its endpoint values are $2(q-1)$ and $q(q-1)/2$. Taking $q = 5, 6, 4$ gives the three asserted gaps. For sharpness set all short entries to two. Generic $x(t) = t^2 b$ gives no long units, while $x(t) = te_1 + t^2 b$ gives exactly $2(q-1)$ long units. The constant-short slice constructions realise both choices. In the third family choose its independent A_1 value to be one or two as appropriate. □

Corollary 5.10. *For each of the ten nonclassical attachment configurations of Theorem 4.1, its numerical image is obtained by fixing the extra weight entries and testing one ordinary or folded root core, with at most one additional independent rank-one coordinate or radial tuple. All these pieces are independent except in the F_4–line configuration. In that configuration their entire mutual compatibility is*

$$n_v = 4 \quad \text{or} \quad \min_{s \text{ short in } F_4} n_s = 2.$$

Proof. The ordinary minuscule attachments have the displayed fixed weight values and an arbitrary admissible ordinary core. The spin-B_5 and spin-B_6 attachments have fixed spin values and one folded core; the $B_4 + A_1$ attachment adds an independent positive A_1 entry. Theorem 6.7 gives the equivalence in the remaining row. □

5.4 Small entries on the other folded supports

On the spin extension of B_5, use the sets of short, long and spin labels in Section 6.2.1.

Corollary 5.11. *Every ordinary positive D_5 root-contact function on the twenty long labels extends to a complete smooth-source spectrum on B by assigning one to the sixteen spin labels and two to the five short labels. Conversely, with these twenty-one entries fixed, the remaining numerical criterion is exactly the ordinary rational root-contact criterion for D_5.*

Proof of Corollary 5.11. The short normalised depths are all one. Their moving coefficient removes all short forms before any positive even long depth. At every subsequent even level the higher-root span tests are precisely those for the long-root subsystem D_5. In the eight-coordinate model of Section 6.2.4, let ξ be the primitive internal A_3 coefficient and let $z(t)$ be a Cartan witness for the long D_5 roots. Then one can use

$$\lambda(s) = s\xi + s^2 e_8 + z(s^4).$$

The same fibre computation gives the stated extension. □

For the diagonal attachment of Section 6.2.4, the only variable entries are the three values on the differences $u_i - u_j$.

Corollary 5.12. *For $N \geq 2$, the number of realisable tuples on any one of these marked supports with every entry at most N is exactly*

$$\frac{3N^2 - N}{2}.$$

The possible numbers of unit entries are four, six and seven; five unit entries are impossible. In particular, the four forced unit entries can occur with all other entries greater than one.

Proof. An A_2 triple either has all entries equal, giving N choices, or has two equal minima a and one value $b > a$, giving $3\binom{N}{2}$ choices. If a difference value is one, the minimum condition forces two or three difference values to be one. Otherwise there are just the four fixed unit entries. □

For B_6 write $x_i(t)$ for the six invariant coordinates and $z(s)$ for the odd coordinate on $t = s^2$ in the construction of Section 6.2.4. The numerical test is given in Section 6.2.1.

Corollary 5.13. *If all six short values are two, the complete criterion reduces to $n_\omega = 1$ on the spin labels and the ordinary rational D_6 contact criterion on the long labels.*

Proof. Choose $z = s$, so every short normalised depth is one. The even coordinates $x_i(t)$ then have arbitrary finite D_6 contacts. Conversely the long contacts in every model are exactly the orders of $x_i \pm x_j$. □

For the F_4 support in Section 6.2.5, the primitive line has value $a = n_v$ and its double has value $b = n_{2v}$.

Corollary 5.14. *Assign one to all mixed labels, five to every short F_4 root, two to every long F_4 root, and $(a, b) = (7, 2)$. The core and the radial tuple separately pass their complete criteria. Their joint values violate the coupling condition. Changing only a to four makes it realisable. Alternatively, keeping $(a, b) = (7, 2)$ and changing all short-core entries to two makes it realisable.*

The calculation is in Section 5.4.1.

5.4.1 The F_4 core and the doubled direction

Give each mixed label value one, each short F_4 root value five and each long root value two. On the line take $(n_v, n_{2v}) = (7, 2)$.

Proof of Corollary 5.14. For the original core all normalised depths equal four, so the intrinsic F_4 tests pass. The radial relation is $7 = 2 \cdot 2 + 3$. The coupling fails because $7 > 4$ and every short entry is five. Taking $n_v = 4$ satisfies the first alternative in Theorem 6.7; keeping $(7, 2)$ and lowering the short-root values to two satisfies the second. More generally, for every $N, M \geq 1$, the core with short values $2N + 1$ and long values N, together with radial values $(2M + 3, M)$ and mixed values one, passes both separate tests and fails the coupling. □

5.5 Quotients and proper flat restrictions

Restriction to a saturated root flat gives the component spectra described in Section 2.2.5. Two examples delimit its use as a realisation test. Quotienting by a unit label can introduce a singularity in the source; checking every proper flat can miss a global linear dependence.

5.5.1 Quotienting by a unit label can destroy realisability

Consider the complete A_3 spectrum with all six entries one. It is realised by the usual small resolution of

$$xy = \prod_{i=1}^{4}(z - a_i t), \qquad a_i \neq a_j.$$

Quotient the root lattice by a simple unit class c_1, omit its zero image, and add multiplicities of labels with the same image. The result on the remaining full A_2 support is

$$(n_{c_2}, n_{c_3}, n_{c_2+c_3}) = (2, 1, 2).$$

Its minimum is attained once, so it fails the necessary and sufficient ordinary A_2 contact criterion. Geometrically, contracting the Atiyah curve introduces a node into the source of the remaining contraction. Thus this operation has left the smooth-source category.

There is an exact statement for this operation in type A. Let H_{ij} be pairwise contact orders and suppose $H_{ij} = 1$. After merging i, j, put $K_k = H_{ik} + H_{jk}$ on the new edges. The resulting array is a contact array if and only if all K_k equal one constant c and all unaffected H_{kl} are at least c. Indeed the first-degree equivalence classes for $H_{ab} > 1$ separate i and j. Two remaining vertices in different such classes have $H_{kl} = 1 < K_k, K_l$, which is forbidden. If two remaining vertices in the same class have unequal depths to its distinguished vertex, their mutual depth is the smaller original depth, strictly smaller than the smaller K. Thus every K must coincide, and the minimum-twice condition imposes precisely the stated lower bounds on the remaining edges. These conditions are also sufficient by checking the triangles through the merged vertex. In particular, for every $r \geq 3$ the full A_r spectrum with all entries one admits no valid unit quotient.

5.5.2 A global obstruction invisible in all proper root-flat restrictions

On the full positive D_4 root support give value two to

$$e_1 - e_2, \quad e_1 + e_2, \quad e_3 - e_4, \quad e_3 + e_4,$$

and value one to the other eight positive roots. The four higher-order roots are linearly independent, so they span every root. The rational test at degree one fails and the full tuple is not realisable.

Every proper rational root-flat restriction, interpreted componentwise, is nevertheless realisable. It contains at most three of the four displayed orthogonal roots. The span of any such proper subset contains no other D_4 roots. Thus its value-two subset is a root flat in the restricted system, and the ordinary rational contact criterion passes. Equivalently choose a degree-one coefficient vanishing precisely on that subset, followed by a generic degree-two coefficient. If the restricted system is reducible, its connected components give separate local contractions. This is the D_4/A_1^4 root-closure obstruction. It proves that the sharp three-block locality of full BC cannot be promoted to a universal locality theorem for arbitrary enhanced supports.

5.6 The arithmetic image on a fixed support

Write $\mathcal{G}(B)$ for the set of all realisable multiplicity functions with complete positive support B. Fixing the residues and the weak order of the surviving contact levels gives an affine monoid for each such choice.

Theorem 5.15 (The effective arithmetic image)**.** *For every finite complete marked positive support B of rank r, the set $\mathcal{G}(B)$ is an effectively constructible finite union*

$$\bigcup_{\nu}\left(b_\nu+\sum_{j=1}^{q_\nu}\mathbb{Z}_{\geq 0}p_{\nu j}\right),\quad b_\nu\in\mathbb{Z}_{>0}^{B},\quad p_{\nu j}\in\{0,1,2,3\}^{B},$$

where each set of generators is linearly independent and $q_\nu\leq 2r$. The cover may overlap. Membership is decidable and acceptance produces an algebraic Cartan witness. For nonempty $\mathcal{G}(B)$ its characteristic generating series is rational, and its box count is eventually quasipolynomial, with a common positive leading coefficient and degree

$$d_B\leq 2r,\qquad d_B\leq |B|-1\quad (r\geq 2).$$

The bound $2r$ is attained in every curve rank. The Zariski closure is a finite union of rational affine subspaces and, for $r\geq 2$, is contained in the union of the coordinate equality hyperplanes.

The classical branch reduction gives an ADE presentation of rank at most $2r+2$, and an exceptional presentation has rank at most eight. Thus every spectrum has a presentation of rank at most $\max(2r+2,8)$.

5.6.1 Free affine monoids and counting

Fix a presentation and an attainable prefix. The surviving root system Ψ contains no internal root. Each of its positive roots has a nonzero positive label. The Coxeter product c fixes that label, so it preserves Ψ^+ and permutes its simple roots. Choose a component Θ from a c-orbit of p components. The return map $\sigma=c^p|_\Theta$ permutes the vertices of its connected ADE diagram. Its order e is one or two, except that triality of D_4 can have order three. By Lemma 2.3, a surviving label meets only one such component orbit. Write ℓ_β for the corresponding σ-orbit lengths. Put $u=s^{m/(pe)}$. Its tail orders k_β contribute

$$n_\beta=b_{\tau,\beta}+\frac{\ell_\beta}{e}k_\beta,\qquad k_\beta>peL/m. \tag{5.1}$$

The component-cycle length cancels. Orthogonal projection and cyclic transport construct the arc on each component orbit independently.

Fix the weak order of the contact levels and their residues modulo e. Coefficient avoidance depends only on these finite choices. If the distinct levels are $k_1<\cdots<k_q$, let d_1 be the least permitted first level and d_j the least positive gap with its prescribed residue. Every sequence in this pattern is uniquely

$$k_j=d_1+\cdots+d_j+e(y_1+\cdots+y_j),\qquad y_i\in\mathbb{Z}_{\geq 0}.$$

Increasing one y_i adds $\ell_\beta\in\{1,2,3\}$ to exactly the labels surviving that gap. Strict nested supports make these generators independent; different component orbits have disjoint label sets. Base-point integrality is checked once. At an attained level k, every root of order greater than k vanishes on the degree-k coefficient, and each root of order k takes a nonzero value there. Thus the spans of successive superlevel sets in $\operatorname{span}\Theta$ strictly decrease, and a component orbit has at most $\operatorname{rank}\Theta$ levels. The dimension of $(\operatorname{span}\Theta)^\sigma$ is the number of vertex orbits. For an involution each orbit has size at most two. In the triality case the four vertices give two orbits. In either case $\operatorname{rank}\Theta\leq 2\dim(\operatorname{span}\Theta)^\sigma$. An invariant vector on one component extends by cyclic transport to a c-fixed vector on its full component orbit. These extensions have disjoint component supports. They lie in the r-dimensional fixed space of c on the ambient Cartan,

since c has no fixed vector on any internal component. Summing over orbit representatives gives

$$q \le \sum \operatorname{rank} \Theta \le 2 \sum \dim(\operatorname{span} \Theta)^{\sigma} \le 2r.$$

Bounded presentations, prefixes and level patterns are all finite. This proves the cover in Theorem 5.15, allowing overlaps.

For effectivity, enumerate the bounded presentations and chamber-basis identifications matching the supplied support. Prefix feasibility is a finite coefficient calculation. Enumerate tail patterns and their finite character tests, or use (2.3) directly for a proposed tuple. An accepted pattern yields a polynomial witness by coefficient avoidance. Alternatively membership in a listed free affine monoid is checked by solving for its unique rational parameters and testing nonnegative integrality.

5.6.2 Counting and sharpness

The set is Presburger definable. Its characteristic generating series is rational, and its box-counting function is eventually quasipolynomial [11]; overlaps must be counted set-theoretically. A monoid of rank q has $\Theta(N^q)$ points in the N-box. Thus the degree of the union's count is the largest nonempty monoid rank. Monotonicity in N forces the leading coefficients on all eventual residue classes to agree.

Each monoid is Zariski dense in its affine span. Theorem 5.8 places the whole image in finitely many equality hyperplanes, so each such irreducible affine span lies in one of them when $r \ge 2$. This gives $d_B \le \min(2r, |B| - 1)$ and density zero.

For sharpness use full BC_r in D_{2r+2} with $Y = s, Z = 2s$ and

$$x_i^+ = s + s^{a_i} + \tfrac{1}{2}s^{b_i}, \qquad x_i^- = -s - s^{a_i} + \tfrac{1}{2}s^{b_i},$$

where $3 \le a_1 < \cdots < a_r$ are odd and $a_r < b_1 < \cdots < b_r$ are even. Fibre sums give

$$n_{2u_i} = b_i/2, \qquad n_{u_i} = a_i + 3, \qquad n_{u_i - u_j} = n_{u_i + u_j} = 1 + \min(a_i, a_j).$$

These entries recover the $2r$ parameters. Choosing the a_i below $N/2$ and the b_i in $(N, 2N]$ gives $\Omega(N^{2r})$ spectra in the N-box. Together with the upper bound this proves sharpness, including $r = 1$.

5.6.3 Counting the contact strata of full BC_r

Fix a partition $\mathcal{P}$ of $\{1, \dots, r\}$ and a subset $\mathcal{A} \subseteq \mathcal{P}$ of at most two attached classes. Let $\mathcal{G}_{\mathcal{P},\mathcal{A}}$ be the full-BC_r spectra whose contact classes and attached flags are exactly this pattern. Write $u = |\mathcal{P} \setminus \mathcal{A}|$ for the number of unattached classes.

Proposition 5.16. *The set $\mathcal{G}_{\mathcal{P},\mathcal{A}}$ is the direct product of its local unanchored $C_{|C|}$ and anchored $BC_{|C|}$ numerical tests; all interactions between distinct classes are fixed to be two. For fixed $r, \mathcal{P}, \mathcal{A}$ its box count satisfies*

$$\#\{n \in \mathcal{G}_{\mathcal{P},\mathcal{A}} : n_\beta \le N\} = \Theta(N^{2r-u}).$$

In the canonical reduced D_{2r+2} model, its first-jet survivor root system has type

$$\bigoplus_{C \notin \mathcal{A}} A_{2|C|-1} \ \oplus \ \bigoplus_{C \in \mathcal{A}} A_{2|C|},$$

with the reversal involution on each summand.

Proof. The independent generic-leading-coefficient gluing in the proof of Theorem 3.1 proves the direct product assertion. A contact class of size m has $2m$ signed coordinates with the same leading coefficient after sign alignment. It contains one additional terminal coordinate exactly when attached. The vanishing root subsystem on these coordinates is therefore A_{2m-1} or A_{2m}, respectively. Distinct leading squares have no roots between them. The internal involution reverses each paired-coordinate system, giving the displayed action.

The A_{2m-1} unanchored contact array has at most $2m-1$ distinct attained depths: every depth causes a strict drop of the span of roots still vanishing. Every depth is at most $2N$ in an N-box. There are only finitely many order patterns for the fixed set of labels, so its box count is $O(N^{2m-1})$. The same argument gives $O(N^{2m})$ in the anchored A_{2m} case.

For matching lower bounds write paired residual polynomials as

$$F_i^+ = O_i + E_i, \qquad F_i^- = O_i - E_i,$$

where O_i is odd and E_i even. In an unanchored class of size m, set $O_m = 0$, choose

$$O_i = s^{h_i} \quad (1 \le i < m), \qquad E_i = s^{k_i} \quad (1 \le i \le m),$$

with odd $3 \le h_1 < \cdots < h_{m-1}$ and even $h_{m-1} < k_1 < \cdots < k_m$. For $m = 1$ omit the h's. The h_i are recovered from the contacts with the last paired block; the k_i are recovered from diagonal multiplicities. Thus this gives $2m-1$ independent integer parameters, all intra-class pair values exceeding two. Choose the odd h_i below $N/2$ and the even k_i in $(N, 2N]$ to obtain $\Omega(N^{2m-1})$ spectra in the box.

For an attached class use $O_i = s^{h_i}$ for all $1 \le i \le m$, with odd $3 \le h_1 < \cdots < h_m$, and again even $k_i > h_m$. Now the terminal multiplicities $a_i = h_i + 3$ recover all h_i, while $b_i = k_i/2$ recover all k_i. The same box choices give $\Omega(N^{2m})$ distinct spectra. Adding the class leading coefficients and its assigned terminal as in the realisation proof supplies the required geometric witnesses. Multiplying the independent local bounds gives degree

$$\sum_{C \notin \mathcal{A}} (2|C| - 1) + \sum_{C \in \mathcal{A}} 2|C| = 2r - u.$$

□

Corollary 5.17. *For every $r \ge 2$, fix any partition into two nonempty attached classes. The corresponding realisable spectra have cross-class entries exactly two and have $\Theta(N^{2r})$ members in an N-box. The full BC_r realisation set also has $\Theta(N^{2r})$ members. Hence the two-anchor sector omitted by the single-anchor criterion contains a proportion bounded away from zero of the realisable spectra in sufficiently large boxes.*

Proof. Here there are no unattached classes, so Proposition 5.16 gives degree $2r$. There are only finitely many contact partitions and attached flags for fixed r, all with degree at most $2r$, while a one-class attached pattern already attains that degree. Thus the full count is $\Theta(N^{2r})$ as well, and the ratio has a positive lower bound. The two classes have no common first anchor, so their entire sector is excluded by the single-anchor rank criterion. □

Corollary 5.18. *Fix $r \ge 1$. For $1 \le k \le r$, let $G_{r,k}(N)$ count the realisable marked full-BC_r tuples with exactly k unit entries and every entry at most N. Then*

$$G_{r,k}(N) = \Theta(N^{2r-k-1}).$$

Consequently, the full-BC_r tuples with at least one unit entry have count $\Theta(N^{2r-2})$ and proportion $\Theta(N^{-2})$ among all realisable full-BC_r tuples in the box.

Proof. Only doubled labels can have value one, so the unit indices are precisely those with $b_i = 1$. In a contact class subtract its common first-order term and write $F_i^+ = O_i + E_i$, $F_i^- = O_i - E_i$, with O_i odd and E_i even. A unit pair has a nonzero degree-two coefficient in E_i; a nonunit pair has zero coefficient there. Every contact between those two kinds of pair therefore has depth two.

Suppose a class has $k_C > 0$ unit pairs and m_C nonunit pairs. Group its nonzero second coefficients up to sign, into $g_C \geq 1$ groups. After orienting pairs, a group of size t gives an ordinary A_{t-1} contact array; equivariance determines its opposite array. It has at most $t-1$ variable contact levels. The nonunit pairs have at most $2m_C$ further levels by the anchored or unanchored rank bound. Thus this class has at most $k_C - g_C + 2m_C$ variable levels. A class without units has at most twice its size. For fixed r only finitely many partitions and sign groupings occur, and every contact depth is at most $2N$. Summing yields $O(N^{2r-k-1})$.

For the lower bound put all indices in one attached class. For its k unit pairs use

$$F_i^+ = s^2 + H_i(s), \qquad F_i^- = -F_i^+(-s),$$

where $H_k = 0$ and $H_i = s^{d_i}$ for $i < k$, with $3 \leq d_1 < \cdots < d_{k-1} < N/2$. The contacts with index k recover these $k-1$ parameters. For each remaining pair use

$$F_j^{\pm} = s^{h_j} \pm s^{e_j},$$

with distinct increasing odd h_j in $[3, N/2)$ and distinct increasing even e_j in $(N, 2N]$. Its entries $a_j = h_j + 3$ and $b_j = e_j/2$ recover another $2(r-k)$ parameters. The first k short values are five, and their doubled values are one. Using $Y = s, Z = 2s$ and common leading coefficient s gives the geometric witness from the factorisation proof. All entries are in the box for sufficiently large N, and the parameter count is $(k-1) + 2(r-k) = 2r-k-1$. This proves the matching lower bound. The final proportion follows from the total count $\Theta(N^{2r})$. □

6 Explicit exceptional models

This section constructs the explicit root-lattice models and proves presentation exhaustion for all individual exceptional families cited in Sections 4 and 5. We group the families by their internal monodromy mechanism: ordinary minuscule attachments with trivial survivor monodromy (Section 6.1), folded root cores with a surviving nontrivial character including spin-B_3, spin-B_5, diagonal-B_3, spin-B_6, $B_4 + A_1$, and the F_4–line coupling (Section 6.2), and the four exceptional rank-two supports including the two-phase doubled-short G_2 configuration (Section 6.3).

The 217 exceptional presentations outside the 27 rows comprise 21 rank-one choices, the three full exceptional root systems, 73 presentations of the eight ordinary weight models, and 120 other presentations of rank at least two with internal roots. In the ordinary models we compute the active internal components and the weights in each fibre. Proposition 6.1 then gives its contribution. For a folded model, take the indecomposable positive roots annihilating the chosen primitive coefficient. Their Gram matrix determines the surviving root system. We also compute the quotient label of each root and the action of c on it, including the sign. These data distinguish the orders of the short and long fibres.

6.1 Ordinary root cores and minuscule weights

For a root set R in its rational span let R^+ be its positive representatives in the chosen chamber. Write $\mathcal{O}_R(n)$ for the ordinary contact condition

$$\beta \notin \operatorname{span}_{\mathbb{Q}}\{\gamma \in R^+ : n_\gamma > k\} \qquad (\beta \in R^+,\ n_\beta = k). \tag{6.1}$$

For these rational forms, extension of scalars to $\mathbb{C}$ preserves all ranks. Lemma 2.2 with $c = 1$ gives a polynomial witness for every such assignment of positive integers.

6.1.1 Components of distinct prime orders

Proposition 6.1. *Suppose $\Phi_I = \bigoplus_j A_{p_j-1}$ for distinct primes p_j, so $m = \prod_j p_j$. Let $R = \Phi \cap I^\perp$. In an external fibre let k_j be the index of its dominant internal weight on the jth factor, with $k_j = 0$ for the zero weight. The fibres with all $k_j = 0$ are singletons labelled by R. Their possible entries are precisely the positive assignments satisfying $\mathcal{O}_R(n)$. For each remaining fibre put $p_* = \max_{k_j \neq 0} p_j$. Its value on every smooth arc is*

$$n_\beta = \frac{1}{p_*} \prod_j \binom{p_j}{k_j}. \tag{6.2}$$

All these values occur together with every allowed ordinary tuple, including when R spans a proper subspace of $I^\perp$.

Proof. Write A_{p-1} in the coordinate hyperplane $\sum x_i = 0$. Pairing an external root with $e_i - e_j$ gives 0 or ± 1. Thus its internal projection is zero or a permutation of the fundamental weight ω_k represented by a k-subset, $0 < k < p$. By Lemma 2.3 the fibre is the full product of these weight orbits. The p-cycle acts freely on the nonempty proper subsets of each fixed size. Hence its product c has orbit length $\prod_{k_j \neq 0} p_j$ in this fibre.

The invariant component of the arc begins at degree m or later. On a factor of order p the first possible degree is m/p. Smoothness makes its coefficient a nonzero vector on the primitive eigenline, with coordinates proportional, in cyclic order, to the pth roots of unity. A nonempty proper subset has an indicator polynomial with both a zero and a unit coefficient in degrees $0, \dots, p-1$. Its sum of roots of unity is nonzero since the minimal polynomial of ζ_p is $1 + X + \cdots + X^{p-1}$. On a nontrivial fibre the factor of order p_* contributes in degree m/p_*. Every other nonzero internal projection first contributes in a strictly higher degree. Thus every root has order m/p_*. This proves (6.2) and gives $(b_\beta, \ell_\beta) = (n_\beta, 0)$. On R the pair is $(0, 1)$, and the depth is mn_β. Only the invariant component evaluates on these roots, giving (6.1).

Conversely choose a polynomial witness $z(t)$ for the ordinary orders in the complex span of R, included in $I_{\mathbb{C}}^\perp$. With a primitive eigenvector ξ_j on each component, take

$$\lambda(s) = \sum_j s^{m/p_j} \xi_j + z(s^m).$$

The coordinates of ξ_j are distinct, so every internal root has its required order. The evaluations just computed are finite on every external root. The geometric input gives the marked contraction. □

6.1.2 The eight integral models

Write $Q = Q(R)$, $P = P(R)$ for the root and weight lattices, and put $[u] = W(R)u \cup (-W(R)u)$. On A_{d-1} use $\omega_k = \sum_{i=1}^k e_i - \frac{k}{d} \sum_{i=1}^d e_i$. Let $h = \frac{1}{2}(1,1,1,1,1,1)$ in the half-spin extension of the even-sum D_6 lattice, v_6, v_7 minuscule fundamental weights of E_6, E_7, and $a = (\omega_1, \omega_1)$ on $A_2 \oplus A_2$. Write $u_p = \omega_1$ on an internal A_{p-1}. The glue column gives the generators to adjoin to $Q \oplus Q(\Phi_I)$ in orthogonal factors. In the last weight column $[u]_{f \mapsto b}$ means a fibre of f roots over each signed label in $[u]$, with forced GV value b. For example, the first row assigns one to each middle-weight label of A_5. Its fifteen positive roots have singleton fibres and carry the ordinary contact values.

$[L:Q]$	Φ	R	L	Glue generators	Additional signed labels	$\lvert B\rvert$
2	E_6	A_5	$Q+\mathbb{Z}\omega_3$	ω_3+u_2	$[\omega_3]_{2\mapsto1}$	25
2	E_7	D_6	$Q+\mathbb{Z}h$	$h+u_2$	$[h]_{2\mapsto1}$	46
2	E_8	E_7	P	v_7+u_2	$[v_7]_{2\mapsto1}$	91
3	E_6	$A_2\oplus A_2$	$Q+\mathbb{Z}a$	$a+u_3$	$[a]_{3\mapsto1}$	15
3	E_7	A_5	$Q+\mathbb{Z}\omega_2$	ω_2+u_3	$[\omega_2]_{3\mapsto1}$	30
3	E_8	E_6	P	v_6+u_3	$[v_6]_{3\mapsto1}$	63
5	E_8	A_4	P	ω_2+u_5	$[\omega_2]_{5\mapsto1},\ [\omega_1]_{10\mapsto2}$	25
6	E_8	A_5	P	$\omega_3+u_2,\ \omega_2+u_3$	$[\omega_3]_{2\mapsto1},\ [\omega_2]_{3\mapsto1},\ [\omega_1]_{6\mapsto2}$	46

Lemma 6.2. *In each row the glued lattice is $Q(\Phi)$, the internal system is parabolic, and projection along Φ_I identifies its quotient with L. For each additional signed label β, let $0\le k_j<p_j$ represent the internal classes paired with β by the glue in $P(A_{p_j-1})/Q(A_{p_j-1})=\mathbb{Z}/p_j$. Then the full root set is*

$$\Phi = R\sqcup\Phi_I\sqcup\bigsqcup_{\beta\ \text{additional}}\left(\beta+\prod_j W(A_{p_j-1})\omega_{k_j}\right),\qquad \omega_0=0. \tag{6.3}$$

The product is taken in the orthogonal internal factors.

Proof. All glue generators have squared length two and pair integrally with the starting lattice. In the last row their mutual pairing is $(\omega_3,\omega_2)=1$. Their classes generate a group of the indicated order, which projects injectively to P/Q and onto the full internal discriminant group. Indeed the class of ω_i for A_{d-1} is i mod d, the half-spin class here has order two, and the exceptional minuscule classes have orders three and two on E_6,E_7. Reflections in the norm-two vectors of the resulting even lattice preserve the lattice. These vectors generate it and form a connected simply laced root system, since the glue roots join all its factors. In ranks $6,7,8$ its determinant, the initial determinant divided by the square of the index, is respectively $3,2,1$. Thus it is the stated ADE lattice. Injectivity of the projected glue classes gives kernel $Q(\Phi_I)$ and image L for projection along the internal space. A generic point in $I^\perp$ therefore exhibits Φ_I as a parabolic subsystem.

In the index-two and index-three rows pair the chosen weight orbit with the standard internal orbit and take both signs. The external squared lengths are $3/2$ and $4/3$, respectively. In index five the external and internal indices are $(2j \bmod 5, j)$ for $1\le j\le4$. In index six the external index is $3i+2j$ mod 6 for internal indices $(i,j)\in\mathbb{Z}/2\times\mathbb{Z}/3$. Every listed vector belongs to the glue lattice and has squared length two by $\omega_k^2=k(d-k)/d$ on A_{d-1} and the indicated minuscule norms. The three signed weight counts in index two are $20,32,56$. Duality fixes the middle weight of A_5, the parity of the six half-spin signs and the unique minuscule node of E_7. In index three the two signs are in different cosets, giving totals $18,30,54$. The exceptional single-orbit sizes $27,56$ follow by orbit–stabiliser from the subdiagrams $D_5\subset E_6$ and $E_6\subset E_7$. The minuscule squared lengths are the corresponding inverse Cartan diagonal entries. Thus the listed vectors together with R and Φ_I count $72,126,240$ in each of the first two groups of three rows. The last two totals are

$$20+20+20\cdot5+10\cdot10=240,\qquad 30+2+6+20\cdot2+30\cdot3+12\cdot6=240.$$

These counts exhaust the roots in each ambient system and prove (6.3). □

Theorem 6.3. *For each row take the union of R with the displayed signed labels in L and choose any ordered chamber marking. A complete positive integral spectrum on this support is realised if and only if its root values satisfy (6.1) and its other values equal the displayed constants.*

Proof. Identity (6.3) supplies the signed labels and the internal orbits to Proposition 6.1. It gives the constants in the table and leaves exactly the singleton depths mn_β on R. For $A_2 \oplus A_2$, (6.1) says that the minimum in each of the two root triples occurs at least twice, independently in the two factors.

To pass to the ordered support, first observe that the inverse moment metric is proportional to the root metric on R by Weyl invariance. For $A_2 \oplus A_2$ the two independent Weyl actions eliminate cross terms and interchange of factors equates the scales. Normalise the roots to have squared length two. The additional squared lengths are $3/2$ in index two and $4/3$ in index three. In index five they are $6/5, 4/5$ in column order, and in index six they are $3/2, 4/3, 5/6$. Thus the longest labels are exactly R. Every signed support symmetry preserves the constants and the rational ranks in (6.1).

For classical presentations of arbitrary ambient rank we use the support formulas of Section 3.2. In rank $r \geq 4$ their positive counts are $r(r+1)/2$ in type A, or $r(r-1)+d$ and r^2+d in the two type-D frames, with $0 \leq d \leq r$. In rank four the sole classical candidate of size 15 consists of the D_4 roots and three pairs $\pm 2e_i$. Its moment matrix in the coordinate frame is $\operatorname{diag}(10, 10, 10, 6)$. The inverse gives three squared lengths $1/5, 4/15, 2/5$, whereas the ordinary 15-label support has two. Size 25 exceeds the rank-four bound. In rank five size 25 requires full B_5 or C_5, with squared-length ratio 2, against $4/3$ in the ordinary row. Size 30 requires full BC_5, whose pair $e_i, 2e_i$ is excluded by the ordinary row's ratio $3/2$. The larger sizes exceed the classical maxima $30, 42, 56$ in ranks five, six and seven.

It remains to compare exceptional presentations. Single roots are Weyl-conjugate. With the first root fixed, the second simple root of an A_2 has inner product -1 with it. By the index-two identity (6.3) these choices form one orbit of the orthogonal Weyl group. Thus the A_2 embeddings are conjugate in each ambient type. In E_8 the orthogonal roots of A_2 form E_6, whose Weyl group is transitive on the extra root of $A_2 + A_1$.

For the counts put $f(T) = |\Phi \cap T^\perp|$. An element w with moving space $\operatorname{im}(1-w) = \mathbb{R}T$ fixes exactly these roots. The first two sets of glue models and successive orthogonal complements give

Φ	$f(A_1)$	$f(A_2)$	$f(A_1^2)$
E_6	30	12	12
E_7	60	30	26
E_8	126	72	60

$$(f(A_3), f(A_2+A_1), f(A_1^3))_{E_8} = (40, 30, 26).$$

For the last triple, extending A_2 to A_3 imposes orthogonality to a minuscule weight in E_6, leaving D_5. Adding an orthogonal root to A_2 leaves A_5. Two orthogonal roots in E_8 leave D_6, in which perpendicularity to a third leaves $D_4 + A_1$. For the Burnside numerator the cycle types of $W(A_1), W(A_2), W(A_3)$ give respectively

$$1 + X, \qquad 1 + 3X + 2Y, \qquad 1 + 6X + 3X^2 + 8Y + 6Z.$$

Multiply for a direct product. A monomial specifies the orthogonal sum T of one A_1, A_2, A_3 for each factor X, Y, Z, respectively, and contributes $f(T)$ with its coefficient. The constant contributes $|\Phi|$. Formula (4.1) gives all exceptional candidates in ranks five through seven by internal type and positive count.

r	E_6	E_7	E_8
5	$A_1 : 25$	$A_2 : 30,\ A_1^2 : 33$	$A_3 : 41,\ A_2 + A_1 : 46,\ A_1^3 : 50$
6	$\varnothing : 36$	$A_1 : 46$	$A_2 : 63,\ A_1^2 : 68$
7		$\varnothing : 63$	$A_1 : 91$

For instance the five cycle types of S_4 give the E_8/A_3 orbit count $(240 + 6 \cdot 126 + 3 \cdot 60 + 8 \cdot 72 + 6 \cdot 40)/24 = 83$, hence $(83-1)/2 = 41$ positive labels.

In rank four E_6 gives $15, 17$ for A_2, A_1^2. For $A_3, A_2 + A_1, A_1^3$ in E_7 the Burnside numerators are respectively $804 + 6x, 504 + 2x, 384 + x$, where in each case x counts roots of a simply laced system of rank at most four, so $0 \le x \le 24$. The resulting counts lie in $[17, 19], [20, 22], [23, 24]$. In E_8 the internal types $A_3 + A_1, A_2^2, A_2 + A_1^2, A_1^4$ give orbit counts at least $2796/48, 2184/36, 1632/24, 1208/16$ by summing just the elements of moving rank at most three. After removing internal orbits their positive counts are at least $29, 30, 33, 36$. The diagram has one standard D_4 subdiagram. The D_4 on the first four coordinates of the usual E_8 model is parabolic since precisely these roots lie in their span. Projection onto the last four gives the 24 roots of D_4, eight vectors $\pm e_i$ and all sixteen half-sign vectors, hence 24 positive labels.

This leaves $A_4 \subset E_8$ for size 25. Number the chain $1, \dots, 7$ with node 8 attached at 3. The complementary sets J of its four-vertex paths are

$$1238, \quad 1278[7], \quad 1678[6], \quad 1267[6], \quad 5678[5], \quad 4567[4].$$

For each bracketed j the longest element on $I \cup \{j\}$ replaces j in J by its opposite in that component, giving an earlier set in the list. It carries the internal root subsystem onto the new one, so Proposition 2.4 transports the entire image. The counts have excluded every other type, and the conjugacies cover every embedding of the surviving types. The symmetry calculation applies in every ordered chamber, so Corollary 2.6 proves necessity there. The glued model and its polynomial arc give sufficiency in the same marking. □

Remark 6.4. In index five each fibre over $[\omega_1]$ has two Coxeter orbits of length five and all ten roots have order one. In index six each fibre over $[\omega_1]$ is one orbit of length six. On $t = s^6$ its mixed weights first evaluate on the degree-two A_2 coefficient, before the degree-three A_1 coefficient. Thus both fibres give multiplicity two. With primitive vectors ξ_3, ξ_2 on these respective factors and v regular on A_5, the arc $s^2\xi_3 + s^3\xi_2 + s^6 v$ realises the index-six tuple with fifteen ordinary units, twenty-five forced units and six forced twos.

6.2 Folded root cores and their attached weights

The external weight orbits can also leave a folded system of roots. The same coefficient test then uses different forms at the even and odd degrees. We first give the six marked supports and their lattices.

6.2.1 The signed supports and the chamber basis

In coordinates $u_1, \dots, u_r$ put $Q_r = \sum_i \mathbb{Z}u_i$, $h_r = \frac{1}{2}\sum_i u_i$, $P_r = Q_r + \mathbb{Z}h_r$, and write

$$S_r = \{\pm u_i\}, \qquad L_r = \{\pm u_i \pm u_j : i < j\}, \qquad W_r = \{\tfrac{1}{2}\textstyle\sum_i \epsilon_i u_i : \epsilon_i = \pm 1\}.$$

The spin supports are $S_r \cup L_r \cup W_r$. The diagonal support adjoins $\pm g$, $g = u_1 + u_2 + u_3$, to $S_3 \cup L_3$. Write $P = \{\pm(u_i + u_j) : i < j\}$ and $N = \{\pm(u_i - u_j) : i < j\}$ in that row. For $B_4 + A_1$ adjoin $\pm w$ and $\Omega = W_4 + \{\pm w/2\}$ to $S_4 \cup L_4$, with w independent, and put $g_w = h_4 + w/2$. For F_4 take $S = S_4 \cup W_4$, $L = L_4$ and $R = S \cup L$. Adjoin $\pm v, \pm 2v$ and $M = S + \{\pm v\}$, with v independent. In each case B is the half on which a generic linear functional is positive. A class in the table denotes either sign. The first pair gives the size of B and the number of positive chamber types up to basis permutation. The reference chamber is specified by its quotient nodes J in E_n, numbered along the chain from 1 to $n-1$ with n at 3. The column $Q(R)$ is

the root lattice used in the odd-degree test.

	$(\lvert B\rvert,\#)$	Λ	n/J	$Q(R)$	relative squared lengths
spin B_3	$(13,2)$	P_3	7/123, 7/125	Q_3	$3_{W_3}, 4_{S_3}, 8_{L_3}$
spin B_5	$(41,7)$	P_5	8/14567	Q_5	$4_{S_5}, 5_{W_5}, 8_{L_5}$
spin B_6	$(68,21)$	P_6	8/134567	Q_6	$2_{S_6}, 3_{W_6}, 4_{L_6}$
diagonal B_3	$(10,5)$	Q_3	6/123	–	$7_{S_3}, 12_P, 15_g, 16_N$
$B_4 + A_1$	$(33,15)$	$Q_4 + \mathbb{Z}g_w$	7/13456	Q_4	$20_{S_4}, 31_\Omega, 40_{L_4}, 44_w$
F_4 with line	$(50,21)$	$P_4 \oplus \mathbb{Z}v$	8/12357	P_4	$15_v, 29_S, 44_M, 58_L, 60_{2v}$

Here the squared length is $\beta^{\mathsf{T}} M_B^{-1}\beta$, where $M_B = \sum_{\beta\in B}\beta\beta^{\mathsf{T}}$. Summing in the displayed coordinates gives $(2r-1+2^{r-3})\mathrm{id}$ in the spin rows, $5\mathrm{id} + gg^{\mathsf{T}}$ in the diagonal row, and respectively $11\mathrm{id}_4 \oplus 5$ and $15\mathrm{id}_4 \oplus 29$ in the last two. After this length check, compare the entire signed set, its positive half and its lattice. In particular $(u_1, \dots, u_{r-1}, h_r)$ is a basis of P_r. Both $Q_r \subset P_r$ and $Q_4 \oplus \mathbb{Z}w \subset Q_4 + \mathbb{Z}g_w$ have index two. A match sends each c_i to the ith atom in the chosen ordering of the positive-label monoid and has determinant ± 1 in lattice bases. For example, in the spin B_3 row arrange $x_i = \ell(u_i)$ in the order $x_1 > x_2 > x_3 > 0$. The only additional wall is $x_1 = x_2 + x_3$. The two ordered bases in the rank-three table below have covolume $1/2$ in Q_3 coordinates.

6.2.2 Insertion on the span of the surviving roots

Use the notation and geometric input of (2.1), and write R_β for its root fibre and $\zeta_j = \exp(2\pi i/j)$. The next result permits additional terms in degrees at which roots outside the surviving span already exit.

Proposition 6.5. *Let $p(s)$ be a centred equivariant polynomial with the prescribed internal orders. Put $\Psi = \{\alpha : \alpha(p) \equiv 0\}$ and identify $T = \operatorname{span}_{\mathbb{C}}\Psi$ with a Cartan subspace by the root inner product. Set $e = \operatorname{ord}(c|_T)$, $q = m/e$ and $V_k = \ker(c|_T - \zeta_e^k)$. For each root outside Ψ suppose*

$$j_\alpha = \operatorname{ord}_s \alpha(p) \;\le\; q\min\{k \ge 1 : \alpha|_{V_k} \neq 0\}, \qquad \min\varnothing = \infty. \tag{6.4}$$

Define $b_\beta = m^{-1}\sum_{\alpha\in R_\beta\setminus\Psi} j_\alpha$ and $l_\beta = |R_\beta \cap \Psi|$. Among arcs with these outside orders, the exact numerical image is given by

$$n_\beta = b_\beta \quad (l_\beta = 0), \qquad d_\beta = \frac{e(n_\beta - b_\beta)}{l_\beta} \in \mathbb{Z}_{>0} \quad (l_\beta > 0). \tag{6.5}$$

The depths in the second formula must also satisfy the following test. Choose $\alpha_\beta \in R_\beta \cap \Psi$. At each attained k and each $d_\beta = k$, require

$$\alpha_\beta|_{V_k} \notin \operatorname{span}_{\mathbb{C}}\{\alpha_\gamma|_{V_k} : d_\gamma > k\}. \tag{6.6}$$

Every acceptance gives one polynomial arc.

Proof. The polynomial p is equivariant and its annihilator contains no internal roots. Lemma 2.3, applied to its coefficient list, puts every nonempty $R_\beta \cap \Psi$ in a single c-orbit. Its orders agree. The product of the exited evaluations over β is invariant under $s \mapsto \zeta_m s$, so b_β is integral.

Projection of an arc to T preserves its evaluations on Ψ and has the form $\eta(s^q)$, with $[u^k]\eta(u) \in V_k$. Formula (2.1) now gives (6.5). Orbit representatives change their degree-k forms only by a nonzero scalar.

Lemma 2.2 on T constructs a polynomial $\eta(u)$ at precisely the prescribed depths under (6.6).

In $\lambda(s) = Cp(s) + \eta(s^q)$ the roots of Ψ evaluate as before. For each other root, (6.4) excludes an earlier term, and the coefficient at j_α excludes just one value of C. One scalar therefore preserves all exits and internal smoothness, also when the inserted arc has a linear term. Necessity of (6.6) follows by evaluating the same common kernel condition on the given arc. □

Corollary 6.6. *Suppose all internal Coxeter numbers equal m. For an internally regular $\xi \in \ker(c-\zeta_m)$ put $\Psi_\xi = \Phi \cap \xi^\perp$, where regularity means $\alpha(\xi) \neq 0$ on all internal roots. Choose such ξ_j so that every Ψ_ξ is contained in some Ψ_{ξ_j} up to an element of W_I commuting with c. Then the whole presentation image is the union of the tests for $p_j = s\xi_j$.*

Proof. After the indicated conjugation, every root outside Ψ_{ξ_j} already exits at order one. The conjugation fixes the quotient labels, and (6.4) holds since the tail is centred. The proposition gives both inclusions. The possible zero sets are read from the root hyperplanes in $\ker(c-\zeta_m)$ after removing the internal hyperplanes. □

For a folded factor R put $R^+ = R \cap B$. For an involution with two survivors over each short label and one over each long label, backgrounds $1, 0$ respectively give

$$d_s = n_s - 1, \qquad d_\lambda = 2n_\lambda, \qquad H_k = \{\gamma \in R^+ : d_\gamma > k\}, \qquad D = Q(R)/Q(R_{\mathrm{long}}). \tag{6.7}$$

The *folded test* below requires all depths positive and, for each $d_\beta = k$, requires $\beta \notin \mathrm{span}_{\mathbb{Q}} H_k$ at even k, and $\bar\beta \notin \mathrm{span}_{\mathbb{F}_2} \overline{H_k}$ in D at odd k. We verify the character forms for each fold. Here $Q(R) = \mathbb{Z}R$. In D_{r+1} change the sign of its last coordinate. A short fibre evaluates by $x_i(u^2) \pm z(u)$, with z odd, and a long fibre by $x_i(u^2) \pm x_j(u^2)$. Thus the even forms are the rational B_r vectors. On the odd line every short form is nonzero and every long form vanishes. These are exactly the ranks of the residues in

$$D = Q_r/Q(D_r), \qquad Q(D_r) = \{x \in Q_r : \textstyle\sum x_i \equiv 0 \pmod 2\}.$$

Thus the odd test says that every odd short depth equals the maximum short depth. The lattice in this test is generated by the core roots. For $r = 3, 5$ the ambient $P_r/Q(D_r)$ is cyclic of order four, with each short root twice a generator. For $r = 6$ it has order four and exponent two.

6.2.3 The numerical rows

In the table Φ/I specifies ambient and internal type, and the number in parentheses is m. The column Ψ gives the roots annihilating the chosen insertion polynomial. On the diagonal row $e = 1$, and elsewhere $e = 2$. In an involutive fold the indicated short and long labels have $(b_\beta, l_\beta) = (1, 2)$ and $(0, 1)$, respectively. The last column gives the background and surviving orbit size for all other labels. For example, a spin B_3 weight has background two and no survivor, while a short root has two survivors of common order $n_s - 1$ in $u = s^{m/2}$. In the last row S_0 is one of the three nonzero residue classes of shorts in $P_4/Q(D_4)$.

Support	Presentation	Surviving system	Other fibres (b_β, l_β)
spin B_3	E_8/D_5 (8)	$D_4 \to B_3$	$W_3 : (2,0)$
	$E_7/(A_3+A_1)$ (4)	$D_4 \to B_3$	$W_3 : (2,0)$
spin B_5	E_8/A_3 (4)	$D_6 \to B_5$	$W_5 : (1,0)$
spin B_6	$E_8/2A_1$ (2)	$D_7 \to B_6$	$W_6 : (1,0)$
diagonal B_3	$E_6/(A_2+A_1)$ (6)	A_2 fixed	$S_3 : (2,0),\ P, g : (1,0),\ N : (0,1)$
	E_7/A_4 (5)	A_2 fixed	$S_3 : (2,0),\ P, g : (1,0),\ N : (0,1)$
$B_4 + A_1$	$E_7/2A_1$ (2)	$(D_5 \to B_4) + A_1$	$\Omega : (1,0),\ w : (0,1)$
F_4, $a = 4$	$E_8/3A_1$ (2)	$(E_6 \to F_4) + A_1$	$M : (1,0),\ v : (4,0),\ 2v : (0,1)$

Support	Presentation	Surviving system	Other fibres (b_β, l_β)
F_4, $a > 4$	$E_8/3A_1$ (2)	$(D_5 \to B_4) + A_2$	$M : (1,0),\ S \setminus S_0 : (2,0)$ $v : (3,2),\ 2v : (0,1)$

Theorem 6.7. *A positive tuple on any of the marked supports above is realised if and only if it satisfies the following conditions on its row.*

1. *The three spin rows require the folded B_r test. On $B_4 + A_1$ use the folded B_4 test and any $n_w \geq 1$. The fixed line has order $2n_w$ in s. Every weight has value two for spin B_3 and one for spin B_5, spin B_6 and $B_4 + A_1$.*

2. *On the diagonal row $n_{u_i} = 2$ for all i, $n_{u_i+u_j} = n_g = 1$ for $i < j$, and the three $n_{u_i-u_j}$ are positive and their minimum occurs at least twice. These are their ordinary depths in $u = t$.*

3. *On the F_4 row all mixed values are one. Use the folded test on the entire F_4 core with $D = P_4/Q(D_4) \simeq \mathbb{F}_2^2$. Put $a = n_v$, $b = n_{2v}$. Then $b \geq 1$ and*

$$a \in \{4, 6, \ldots, 2b+2\} \cup \{2b+3\}, \qquad a = 4 \quad \text{or} \quad \min_{s \in S} n_s = 2. \tag{6.8}$$

   In particular for $a > 4$ the common order over v after its six first-order exits is $a - 3$, and the order over $2v$ is $2b$.

Every accepted tuple is realised by a centred polynomial Cartan arc with the given ordered marking.

6.2.4 The spin and diagonal fibres

Use the E_8 roots $\pm e_i \pm e_j$ and $\frac{1}{2}\sum_{i=1}^{8} \epsilon_i e_i$ with $\prod_i \epsilon_i = 1$. Take E_7 perpendicular to $e_7 + e_8$. The listed external roots generate the projected lattice. A generic functional on this lattice vanishes precisely on the internal roots when pulled back to the ambient Cartan. The incident chamber construction in Proposition 2.4 therefore gives its ordered marking, including chambers separated by a weight wall.

One primitive coefficient. In E_8 take internal D_{j+1} in the last $j+1$ coordinates, with $(r, j) = (3, 4)$ or $(5, 2)$. A Coxeter element acts by a negative j-cycle and negates the last coordinate. On the cycle its primitive eigenvector ξ has, up to order and signs, entries $1, \zeta_{2j}, \ldots, \zeta_{2j}^{j-1}$. These are rationally independent since $X^j + 1$ is cyclotomic. Smoothness forces a nonzero multiple of ξ in degree one. All roots meeting the cycle exit there. The roots over a spin weight have 2^j permitted internal sign choices. Over u_i, exactly $2j$ roots exit and $e_i \pm e_8$ survive. Each long root is a singleton. The projections are exactly S_r, L_r, W_r and generate P_r. The roots on the first r axes and the last axis form D_{r+1} with its fork involution. Proposition 6.5 with $p = s\xi$ gives the first E_8/D_5 and E_8/A_3 entries, since $2^j/(2j)$ is two and one, respectively.

For the other spin B_3 model use $D_3 = A_3$ on axes $4, 5, 6$ of E_7 and the internal root $w = e_7 - e_8$. Here $j = 2$ and the primitive coefficient is on axes $4, 5$. The last two signs of each half-integral root are opposite, giving eight first-order exits per spin weight. Over each short root four roots exit and $e_i \pm e_6$ survive. The mandatory A_1 term has degree two and is orthogonal to the whole surviving D_4. Taking $p = s\xi + s^2 w$ therefore gives the same folded image, including depth one in $u = s^2$. The four marked D_4 systems have the stated fork involution in both chambers. The labelled fibre computation is included in Appendix B.3.4.

Two internal roots of order two. For $r = 6$ in E_8 and $r = 4$ in E_7 take the orthogonal internal roots $e_{r+1} \pm e_{r+2}$. Projection gives the stated spin or $B_4 + A_1$ lattice. In E_7 the extra fixed root is $w = e_7 - e_8$. The opposite last signs project the weights to $W_4 + \{\pm w/2\}$. Write

y, z for the two odd coordinates and $x_i(t)$ for the fixed ones, with $t = s^2$. Smoothness is exactly $\operatorname{ord}_s(y - z) = \operatorname{ord}_s(y + z) = 1$. At least one of y, z has order one. The internal reflection interchanging them fixes the curve lattice, so take y active. Each weight fibre has two roots with first coefficients proportional to those of $y + z$ or $y - z$, so has value one. The short fibre has two first-order exits $x_i \pm y$ and the pair $x_i \pm z$. Even and odd orders cannot cancel. Thus

$$n_{u_i} = 1 + \min\{\operatorname{ord}_s x_i(s^2), \operatorname{ord}_s z\}, \qquad n_{u_i \pm u_j} = \operatorname{ord}_t(x_i \pm x_j).$$

Here and below a zero coordinate series has order infinity. If both odd coordinates are active in degree one, every short depth is one. Conversely, for any folded tail take $y = Cs$ with $C \neq 0, \pm[s]z$. This gives the covering in Corollary 6.6 with survivor D_{r+1} and, in E_7, an orthogonal fixed A_1. Give the latter root the evaluation t^{n_w}. This proves the row even when its tail and its smoothness conditions use the same degree.

The diagonal and its fixed root system. Use $Q(A_2)^3 + \mathbb{Z}(w_1, v_1, z_1)$, with w_i, v_j, z_k the standard weight triples in three orthogonal factors. Each triple sums to zero. The norm-two vectors of the lattice are the eighteen factor roots and the 54 vectors $\pm(w_i, v_j, z_k)$, giving E_6. Take the first A_2 and $\alpha = v_1 - v_2$ internal. Under projection v_1, v_2 have image $q_0 = -v_3/2$. Put $u_k = q_0 + z_k$, so $g = 3q_0$. Over u_k the mixed roots use $j = 1, 2$, giving six roots. Over $-(u_i + u_j) = -2q_0 + z_k$ there are three. The two roots over g are $v_1 - v_3, v_2 - v_3$. Each $u_i - u_j$ lifts uniquely to $z_i - z_j$. This gives the full signed support in Q_3. On the sixfold cover take $p = s^2\xi + s^3\alpha$ with ξ a primitive first-factor eigenvector. The three $w_i(\xi)$ are a common nonzero scalar times $1, \zeta_3, \zeta_3^2$. Here and in the E_7 spin B_3 model the internal components have different Coxeter numbers. Proposition 6.5 applies directly since the tail starts no earlier than the last required internal degree. Smoothness forces both internal activations. Hence the four fibre contributions are $6 \cdot 2/6$, $3 \cdot 2/6$, $2 \cdot 3/6$ and an ordinary order in t, respectively.

For an independent calculation of the competitor, take A_4 in E_7 on the differences of the first five coordinates. Write q_0 for their common projection, $f = e_6$, $w = e_7 - e_8$, and take

$$u_1 = 2q_0, \qquad u_2 = (-q_0 + f + w)/2, \qquad u_3 = (-q_0 + f - w)/2.$$

The vector roots project, up to sign and zero, to $2q_0, q_0 \pm f, w$. With j minus signs among the first five, the half-integral roots project to $((5 - 2j)q_0 + (-1)^{j+1}f \pm w)/2$, with multiplicity $\binom{5}{j}$ for each last sign. Grouping opposite labels gives fibre sizes $10, 5, 5, 1$ on S_3, P, g, N. The five-cycle has irreducible cyclotomic polynomial on the internal A_4. A rational root kills its required primitive coefficient exactly when perpendicular to A_4. These are the six roots over N, fixed by c. Every other external root exits at order one on the fivefold cover. In both models the tail is therefore ordinary A_2 in t. Its three forms are $X, Y, X + Y$, any two independent. Condition (6.6) is precisely the repeated-minimum condition. All fibres over the three short directions have already exited, leaving only the fixed A_2 of differences. The two models give equal marked images. Section 6.2.6 gives the five chamber bases and compares all compatible presentations.

6.2.5 The three internal coefficients on the F_4 row

Take $e_5 - e_6, e_7 - e_8, e_7 + e_8$ internal in E_8. Projection sends e_i to u_i for $i \leq 4$, both e_5, e_6 to v, and e_7, e_8 to zero. Write an equivariant arc on $t = s^2$ as

$$(x_1(t), \dots, x_4(t), A(t) + P_1/2, A(t) - P_1/2, (P_2 + P_3)/2, (P_2 - P_3)/2).$$

Here $P_i(s)$ are odd, and smoothness means that all three have order one. Write $S_{23} = S_4$, and let S_{12}, S_{13} be the half-weights in W_4 with sign product $-1, +1$, respectively. These are

the three nonzero classes of $P_4/Q(D_4)$ on short roots. For $\delta \in S_{ij}$ and $\{i,j,k\} = \{1,2,3\}$ the complete fibre evaluations are

label	evaluations
$\lambda \in L$	$\lambda(x)$
δ	$\delta(x) + (\pm P_i \pm P_j)/2$
$\delta + \epsilon v$	$\delta(x) + \epsilon A \pm P_k/2 \quad (\epsilon = \pm 1 \text{ fixed})$
v	$A + (\pm P_1 \pm P_2 \pm P_3)/2$
$2v$	$2A.$

(6.9)

For example, a core half-weight requires opposite signs on axes five and six, leaving $\pm P_1/2$. The parity of its first four signs chooses P_2 or P_3 on the last two axes. For a mixed half-weight the signs on five and six agree, interchanging those two choices. The vector roots give the other entries by addition and subtraction. This proves the signed-support and lattice assertion, and every mixed entry is one.

Let $c_i' = [s]P_i$. We may change the signs of the three series independently in (6.9). Choose equal representatives within each class of the nonzero c_i' modulo sign. Then $c_i' + c_j' \neq 0$ for every pair. Two of the four roots over $\delta \in S_{ij}$ exit in degree one. The other two have equal orders, giving

$$n_\delta - 1 = \min\{\operatorname{ord}_s \delta(x), \operatorname{ord}_s(P_i - P_j)\}, \qquad 2n_\lambda = \operatorname{ord}_s \lambda(x) \quad (\lambda \in L). \tag{6.10}$$

The even forms are the rational F_4 vectors. At odd degrees the three difference forms $P_i - P_j$ on the plane $\sum P_i = 0$ have every pair independent and their triple dependent. With parallel copies in each S_{ij} and zero on L, their ranks are precisely those in $P_4/Q(D_4)$. Consequently (6.6) gives the stated folded test. Conversely, that test constructs four even series x_i and three odd polynomials T_i with $\sum T_i = 0$ and the prescribed depths in (6.10) with $P_i = T_i$. The differences are unchanged by a common odd translation.

Lemma 6.8. *For every smooth arc in* (6.9)*, the radial values satisfy* (6.8)*.*

Proof. The root over $2v$ gives $b = \operatorname{ord}_t A$. Pairing opposite sign choices in the v fibre gives

$$a = \sum_{\epsilon \in \{\pm 1\}^3/\{\pm 1\}} \min\{2b, \operatorname{ord}_s(\epsilon_1 P_1 + \epsilon_2 P_2 + \epsilon_3 P_3)\}. \tag{6.11}$$

At most one of these four signed sums loses its linear term. Indeed, representatives of any two distinct sign pairs can be made to differ in just one position, and $c_i' \neq 0$. Thus $a = 4$, or $a = 3 + \min(2b, j)$ with $j \geq 3$ odd or infinite. This gives the radial list in (6.8). If all short entries exceed two, (6.10) forces $c_1' = c_2' = c_3'$. Every signed sum then has nonzero linear term, giving $a = 4$. □

For $a = 4$, set $A = t^b$ and $P_i = T_i + Cs$. The common direction $P_1 = P_2 = P_3 = s$, $x = A = 0$, has annihilator $E_6 + A_1$. Indeed the core survivors consist of the D_5 roots on axes $1,2,3,4,8$ and the 32 half-integral roots with $\epsilon_5 = -\epsilon_6 = -\epsilon_7$. Each meets D_5 nontrivially and their components perpendicular to D_5 span one line. This gives the irreducible rank-six system of 72 roots. The remaining two roots are $\pm(e_5 + e_6)$. For a generic C every P_i, every $P_i + P_j$, and every signed triple sum has nonzero linear term. The coefficients of C in the latter sums are ± 1 or ± 3. Thus all internal orders, mixed entries and $a = 4$ hold together, while the folded depths and b are prescribed independently.

Suppose now $a > 4$ and the core passes with minimum short entry two. A short root of depth one cannot avoid two independent higher classes in $P_4/Q(D_4)$. Nor can it avoid a higher root of its own class. Thus the shorts of depth greater than one are either empty or exactly one class S_{ij}. Choose this pair i,j when it occurs, and any pair otherwise. Its four

short directions are orthogonal and their signed pair sums give L. On $S_{ij} \cup L$ the test is the folded B_4 test, with residue lattice $(Q(D_4) + \mathbb{Z}\delta)/Q(D_4)$ for any $\delta \in S_{ij}$. Take the resulting even $x(t)$ and one odd series Z for this class. The four independent short coordinates identify its fixed Cartan with the given x-space. With k the remaining index, put

$$P_i = Cs + Z + \Delta/3, \qquad P_j = Cs - Z + \Delta/3, \qquad P_k = -2Cs + \Delta/3, \qquad A = t^b. \quad (6.12)$$

The direction $(1, 1, -2)$ in these three coordinates has annihilator $D_5 + A_2$. The long roots and the two survivors over each short in S_{ij} give D_5 on the four fixed coordinates and the difference direction. The two survivors over v and the singleton over $2v$, with their negatives, give A_2 on the A and sum directions. These factors are orthogonal since the internal quadratic form in the P-coordinates is $\frac{1}{2}\sum P_i^2$. In particular the radial A_2 has a nontrivial involution. In (6.12) the two moving forms on these factors are $(P_i - P_j)/2 = Z$ and $P_1 + P_2 + P_3 = \Delta$.

All roots outside this annihilator have a nonzero linear coefficient for a common generic C. Hence the other short entries are two, the chosen folded tail is unchanged, and (6.11) gives $a = 3 + \min(2b, \operatorname{ord}_s \Delta)$. Take $\Delta = 0$ when $a - 3 = 2b$. In the remaining cases $a - 3$ is odd and lies between 3 and $2b - 1$, so take $\Delta = s^{a-3}$. This constructs every tuple in the second stratum, also for $b = 1$. It uses the specified short class in its original marking. Equivalently, the three choices are permuted by simultaneous diagram triality on the two D_4 factors of E_8. It preserves their diagonal discriminant gluing and on the last factor fixes the highest root $e_5 + e_6$, twice the projected line generator. Both strata belong to the same sectional model.

Remark 6.9. The values $n_s = 3$ for every short root, $n_\lambda = 1$ for every long root and $(a, b) = (5, 1)$ pass the separate core and radial tests. All the core depths are two. The equalities of leading coefficients above force $a = 4$. With that change, $x = t(1, 2, 4, 8)$, $A = t$ and $P_1 = P_2 = P_3 = s$ give precisely these entries and all mixed entries one. Keeping x, A and choosing $(P_1, P_2, P_3) = (s, 2s, -3s)$ gives $(5, 1)$ with every short entry two. Thus the $E_6 + A_1$ insertion forces $a = 4$, and each $D_5 + A_2$ insertion forces two short classes to have value two.

6.2.6 Comparison over the marked support

Completion of the proof of Theorem 6.7. In a classical sectional presentation the block coordinates give at most $r(r-1)$ positive pair directions and r axes, each axis carrying at most its single and its double. Type A_r gives $r(r+1)/2$ labels. These counts follow from Section 3.2. The reduced rank-three supports here have ten and thirteen rays, exceeding nine. In ranks five and six the sizes in question exceed 30 and 42 respectively.

For exceptional sections use (4.1) and the rank-five and rank-six counts in the proof of Theorem 6.3. At rank five the sizes $33, 41, 50$ select $E_7/2A_1$, E_8/A_3 and $E_8/3A_1$, respectively. In rank six the size 68 selects $E_8/2A_1$.

For the selected orthogonal pairs in E_7, E_8 and triples in E_8 there is one Weyl orbit. Fix one root and use the transitive Weyl group of its perpendicular system D_6 or E_7, respectively. After fixing two in E_8 the perpendicular system is D_6. The A_3 parabolics in E_8 are conjugate as well. Number the chain 1 through 7 with 8 attached to 3. Three-node paths slide through four-node chains by the opposition in A_4. This connects all paths in the long chain, and connects 238 to 123 and 348 to 345. Thus the coordinate models above cover every embedding left by the counts. For each chamber the ordered simple projections described above give its identification. Projecting the roots onto each J and comparing the complete signed supports gives the four lists in ranks five and six by Appendix B.2.

In rank three, $(|B|, \det M_B)$ in a lattice basis is $(13, 864)$ for spin B_3 and $(10, 200)$ for the diagonal. The other exceptional thirteen-label signed family has determinant 756. For a positive support in chamber coordinates put $s_i = \sum_{\beta \in B} \beta_i$. The projection and integral basis comparison in Appendix B.2 give the following list. Subscripts are coordinate orders taking a repetition to the first positive support in that line. The last column gives the ordered

projected simple roots for the first J in that line. Put $p_{ij} = u_i + u_j$, $d_{ij} = u_i - u_j$, and $h = (u_1 - u_2 - u_3)/2$.

	sorted s_i	J (E_6)	J (E_7)	J (E_8)	ordered atoms
spin	$10, 18, 24$		123	567_{321}	$(d_{23}, d_{12}, -h)$
	$10, 14, 18$		125	167_{321}	(d_{23}, u_3, h)
diagonal	$6, 8, 10$	$146,\ 256_{213}$	167_{231}		$(p_{13}, u_2, -p_{23})$
	$6, 8, 14$	$136,\ 356_{213}$	126_{321}		$(g, -u_2, d_{23})$
	$6, 10, 10$	$124,\ 245_{321}$	567_{213}		$(d_{12}, p_{23}, -u_3)$
	$6, 10, 12$	$123,\ 345_{321}$	456_{321}		(d_{12}, d_{23}, u_3)
	$8, 10, 14$	$236,\ 346_{213}$	127_{321}		$(g, -p_{13}, d_{12})$

The finite calculation projects the positive roots onto each of the $\binom{6}{3} + \binom{7}{3} + \binom{8}{3} = 111$ triples J and compares the full sets in the quotient bases. For the last column one can order the atoms by the fibre sizes over the simple projections, using their highest-root coefficients to break ties. For example, at $E_6/146$ the sizes are $3, 6, 3$ and the maxima are $1, 2, 2$. Appendix B.2 gives the root formula and the integral basis test. Within each of these two signed families, the listed parabolics of a given ambient type are joined by chamber moves. Each move applies opposition on $I \cup \{j\}$ and conjugates the internal parabolic. Conjugating further inside the new parabolic aligns the Coxeter elements and fixes all quotient labels. The resulting signed lattice map transports every fibre order. Pull back the target chamber functional along this map and choose that positive half in the model. Its atoms in the specified order then map to the given curve basis. Across ambient types the separate fibre calculations above give equal images for spin B_3 and for the diagonal. Finally the length classes, the rational spans, the residue ranks and the minimum conditions are preserved by every integral identification of the given support and chamber. Applying the inverse marking to the constructed arc gives exactly the prescribed complete spectrum by (2.1). □

6.3 Contact orders on four marked supports of rank two

The rank-two cases have four different outcomes: two independent choices, one compatibility equation, separation of two rays by their Fourier degrees, and a union of two disjoint images. We use $\operatorname{ord}(0) = +\infty$ for individual scalar series. Signs select the positive representative in the input chamber. We use (2.1) and Corollary 2.6.

6.3.1 Integral markings and root fibres

Each line specifies the whole support by taking both signs of its vectors. In the first three lines the two axes are a lattice basis.

$\mathcal{B}_5$	$a,\ b, 2b,\ a \pm b$
$\mathcal{B}_8$	$b,\ a \pm b,\ 2b,\ a,\ 2a,\ a \pm 2b$
$\mathcal{B}_9$	$w, 2w, 3w,\ q, 2q,\ q \pm w,\ q \pm 2w$
$\mathcal{G}$	$S,\ 2S,\ L, \quad S = \{(0,1), (1,1), (1,2)\}, \quad L = \{(1,0), (1,3), (2,3)\}.$

For one representative of each sign-pair put $M = \sum_v vv^{\mathsf{T}}$. In the stated coordinates the first three matrices are $\operatorname{diag}(3, 7)$, $\operatorname{diag}(9, 15)$ and $\operatorname{diag}(24, 9)$. For $\mathcal{G}$ the two rows are $(16, 24)$ and $(24, 48)$. Thus the determinants are $21, 135, 216, 192$. An integral identification preserves the pairings for M^{-1}. After using them to screen basis images, check determinant ± 1, equality of the entire signed sets and equality of the ordered positive-monoid atoms with (c_1, c_2). On $\mathcal{B}_5$ the doubled primitive direction fixes b up to sign, and a is the midpoint of the three labels in one nonzero coset modulo $\mathbb{Z}b$. On $\mathcal{B}_8$ the pair $a \pm 2b$ distinguishes the two doubled axes. On $\mathcal{B}_9$ only the w axis contains a triple. On $\mathcal{G}$ doubling identifies the short directions and hence

the long ones. Its chambers are those of the reduced G_2 arrangement, and the Weyl group preserves the doubled short directions. Independent axis signs on the first three supports exchange only mixed labels of equal prescribed value. On the fourth, the tests below are invariant under the induced permutations of the three sets.

The inequalities in the table concern values of a regular functional, after choosing signs which make both axes positive. Exceptional vertices are numbered $1, \dots, n-1$ along the chain, with n joined to 3. An entry ij sends the quotient simple roots at i, j, in that order, to the two listed atoms. If the input orders those atoms oppositely, use the opposite order on both sides.

	chamber	atoms	E_6	E_7	E_8
$\mathcal{B}_5$	$a > b$	$(a-b, b)$	$14, 52$	67	–
	$b > a$	$(a, b-a)$	$26, 46$	71	–
$\mathcal{B}_8$	$a > 2b$	$(a-2b, b)$	–	13	75
	$2b > a > b$	$(a-b, 2b-a)$	–	32	56
	$b > a$	$(a, b-a)$	–	25	61
$\mathcal{B}_9$	$q > 2w$	$(w, q-2w)$	–	–	27
	$2w > q > w$	$(q-w, 2w-q)$	–	–	28
	$w > q$	$(w-q, q)$	–	–	68
$\mathcal{G}$		$((1,0),(0,1))$	–	53	15

These are the rank-two rows $E_6/14, E_7/25, E_8/27, E_7/35$ of the enumeration in Appendix B.2. Its three-leg root formula gives the full positive sets by coordinate deletion. For example, for $E_8/68$ the simple labels at 6 and 8 have respectively 12 and 20 lifts, which distinguishes $w-q$ from q even though both coordinate maxima are three. The orbit count (4.1) also excludes candidates. Here B is the positive support. Both nine-label families pass the same cardinality test, and their moment determinants separate them. The bracketed opposition moves in the chamber table conjugate the internal subsystems by integral Weyl maps. Internal Weyl conjugation then aligns the Coxeter elements and fixes all quotient labels. This transports the entire arc in each ambient type. The classical frames of Section 3.2 have at most six positive labels in rank two. For $\mathcal{B}_5$ their full list consists of a singleton, one permutation block of size $d \geq 2$ and a terminal D_{e+1} with $e \geq 1$, in D_{d+e+2}. Its quotient is $\mathbb{Z}a \oplus \mathbb{Z}b$.

Use the label order in the first display of this subsection. Each entry for a mixed pair or a set $S, 2S, L$ applies to each of its labels. The vector b_0 gives the sum of the exited orders divided by m, and r counts the roots still to be evaluated. For the classical line we start before choosing any coefficient and abbreviate $D = D_{d+e+2}$, $g = e+1$, $m_D = \operatorname{lcm}(d, 2e)$.

model	m	b_0	r	construction and image
5, D	m_D	0	$(2g, 2dg, \binom{d}{2}, d)$	branches, full in D_5
5, E_6	4	$(2,2,0,1)$	$(0,4,1,0)$	A_3 differences, $n_a = 2$
5, E_7	6	$(2,3,0,1)$	$(3,2,1,0)$	$A_2 + 3A_1$, $n_a \geq 3$
8, E_7, E_8	$4, 8$	$(4,2,1,2,0,0)$	$(0,0,2,4,1,2)$	D_4 coordinates, equal images
9, E_8	6	$(4,2,0,3,0,2,1)$	$(6,3,2,2,1,0,0)$	A_5 sums, independent rays
G, E_7	6	$(4,1,0)$	$(0,0,4)$	three quadratic blocks, $n_s = 4$
G, E_8	6	$(5,1,1)$	$(0,0,2)$	exchanged A_2 factors, $n_s = 5$

(6.13)

For $\mathcal{B}_5, \mathcal{B}_8, \mathcal{B}_9$ the exceptional exits in (6.13) occur at s-degree one. For the five-label counts, project the positive roots of $E_6/14$ and $E_7/17$ and restrict their forms to the first primitive eigenspace. Over the five-label support this gives the A_3 differences with an internal A_1 in E_6. In E_7 it gives two exchanged simple roots of A_2 and three orthogonal roots in a single cyclic orbit. For $\mathcal{B}_8$, project the surviving D_4 roots with $\pi(e_1) = \pi(e_2) = a$, $\pi(e_3) = 2b$ and $\pi(e_4) = 0$. The root $e_1 - e_2$ is internal. For $\mathcal{B}_9$ the minuscule products in Section 6.3.5 give the counts. For $\mathcal{G}$ the roots exit at two distinct degrees; see Section 6.3.6 and Appendix B.3.4. The classical counts are the roots $\pm e_i \pm e_j$ on the indicated coordinate blocks.

At each initial degree used for the exceptional exits there is a single primitive eigenline. Smoothness requires a nonzero vector on it, so rescaling leaves all the exits unchanged. In a component of order h one can take the first nonzero vector $\sum_{j=0}^{h-1} \zeta_h^{-j} c_K^j \alpha_i$ as α_i runs over its simple roots, at degree m/h. For a root column α from the three-leg formula and such a column ξ, the exit test is $\alpha^{\mathsf{T}} C_\Phi \xi \neq 0$ in $\mathbb{Q}(\zeta_h)$, with C_Φ the Cartan matrix. The roots annihilating the initial terms span a c-stable space. Orthogonal projection to it preserves their evaluations. Conversely, add an equivariant polynomial in that space to the initial terms. If its degrees exceed the degrees of the exits, those exits persist. Internal roots still in the span must have their prescribed orders there.

6.3.2 Three interpolation calculations

For positive integers P put

$$D(P) = \{1, 3, \dots, 2P-1\} \cup \{2P\}, \qquad R(P) = 3 + D(P) = \{4, 6, \dots, 2P+2\} \cup \{2P+3\}.$$

For $t = u^2$ write $v(F) = \operatorname{ord}_u F$. If $\operatorname{ord}_t p = P$ and O is odd in u, then $v(p+O) = \min(2P, v(O))$. Its image is $D(P)$, with witnesses $O = u^k$ for odd $k < 2P$ and $O = 0$ for $k = 2P$.

Lemma 6.10. *Fix $P \geq 1$. Let $p, r \in t\mathbb{C}[[t]]$ with $\operatorname{ord}_t p = P$, and let $Q, H \in \mathbb{C}[[t]]$ with $Q(0) \neq 0$. Put $d_\pm = v(p + u(Q \pm H))$ and $v_0 = v(r + uH)$. The finite pairs*

$$A = 2 + d_+ + d_-, \qquad B = 1 + v_0$$

are precisely those with $A \in R(P)$, $B \geq 2$ and $(A-4)(B-2) = 0$. Every pair has a polynomial witness with $p = t^P$ and $Q = 1$.

Proof. At least one $d_\pm$ is one, and the other belongs to $D(P)$. If $A > 4$ then $H(0) = \pm Q(0) \neq 0$, so $v_0 = 1$. For $A = 4$ put $j = B - 1$. Use $(r, uH) = (t, 2u)$ for $j = 1$, $(u^j, 0)$ for even j, and $(0, u^j)$ for odd $j > 1$. For $A > 4$ take $r = 0$ and put $k = A - 3$. Set $uH = u - u^k$ if k is odd, and $uH = u$ if $k = 2P$. These choices give the indicated orders also for $P = 1$. □

Lemma 6.11. *Let $x_j(s) = f(\zeta_m^j s)$ for $0 \leq j < m$, with $\sum_j x_j = 0$. For divisors $r > 1$ of m, prescribe polynomials $G_r = \sum_{l=0}^{r-1} x_{lm/r}$. They are simultaneously attainable exactly when G_r has only degrees h divisible by r and not by m, and $[s^h]G_r/r$ agrees for all prescribed r dividing h. The coefficient of s in f can be chosen freely for $m > 1$.*

Proof. The coefficient multiplier on s^h in the sum is $\sum_{l=0}^{r-1} \zeta_r^{lh}$, equal to r if $r \mid h$ and zero otherwise. Trace zero removes the multiples of m. Assign the common coefficient to f at each prescribed degree. All unused coefficients can be set to zero. In particular two divisors with least common multiple m have independent sums. □

Lemma 6.12. *Fix $d \geq 2$. On $t = u^d$ take finitely many blocks $\{x_i(\zeta_d^j u) : 0 \leq j < d\}$ with $x_i(0) = 0$, $x_i'(0) \neq 0$. For distinct blocks with finite contacts, let N_{ij} be the sum of the d^2 difference orders divided by d. Their joint image consists of $N_{ij} = d - 1 + h_{ij}$, where the h_{ij} are positive symmetric integers whose minimum occurs at least twice on each triangle. Polynomial blocks with total trace zero suffice.*

Proof. Rotate each block so that slopes in the same μ_d-orbit agree. Of the d differences $x_i(u) - x_j(\zeta_d^k u)$, every one with $k \neq 0$ has order one. The normalised sum is thus $d - 1$ plus $\operatorname{ord}_u(x_i - x_j)$, which satisfies the triangle condition. Conversely, at each attained depth h, give equal coefficients of u^h to indices in the same class for $h_{ij} > h$ and distinct coefficients to different classes. The triangle condition makes these equivalence classes (with the diagonal included), and the resulting polynomials have the prescribed difference orders. Add a common Cu to them. Finitely many C give a zero slope or a slope ratio in $\mu_d \setminus \{1\}$. Choose any other C and subtract the mean of all the translates from every coordinate. The mean is invariant and has order at least d if nonzero. All contacts and the nonzero slopes persist. □

6.3.3 The four numerical rules

Theorem 6.13. *In the ordered markings described in Section 6.3.1, the following conditions are necessary and sufficient.*

1. $\mathcal{B}_5$. *On each of the two marked chambers, the complete image is $n_{a\pm b} = 1$, $n_a \geq 2$, $P = n_{2b} \geq 1$, $n_b \in R(P)$. The D_5 image is the whole set. The E_6 and E_7 images are its respective slices $n_a = 2$ and $n_a \geq 3$.*

2. $\mathcal{B}_8$. *Put $P = n_{2a}$, $A_0 = n_a$ and $B = n_{2b}$. In all three chambers the E_7 and E_8 images coincide. Their common image is given by*

$$\begin{array}{llll} & n_b = 4, & n_{a\pm b} = 2, & n_{a\pm 2b} = 1, \\ P \geq 1, & A_0 \in R(P), & B \geq 2, & (A_0 - 4)(B - 2) = 0. \end{array}$$

3. $\mathcal{B}_9$. *In every listed chamber $n_{q\pm 2w} = 1$, $n_{q\pm w} = 2$, and the two radial tuples range independently over*

$$(n_w, n_{2w}, n_{3w}) \in \{(6,3,1)\} \cup \{(7,k,1) : k \geq 4\}, \qquad n_{2q} = b \geq 1, \quad n_q \in R(b).$$

4. $\mathcal{G}$. *The doubles have $n_{2s} = 1$ for every $s \in S$. The three n_s have a common value ϵ, equal to 4 in E_7 and 5 in E_8. In each phase the three long entries are arbitrary integers at least two whose minimum occurs at least twice. The two images are disjoint.*

6.3.4 One doubled ray and the classical competitors

Proof of the five-label row of Theorem 6.13. The shear in Section 3 preserves the five labelled intersection orders and sends every classical presentation to the D_5 model ($d = 2, e = 1$). In its notation take $t = u^2$, one singleton $x(t)$, one pair $p(t) \pm uQ(t)$ with $Q(0) \neq 0$, and odd terminals Y, Z with $v(Y \pm Z) = 1$. At least one terminal has order one. The mixed values are one and $P = \operatorname{ord}_t p$. The two terminals have distinct leading squares whenever both are active. At most one of them can match the leading square of uQ. Thus the four contributions to n_b give $3 + k$, $k \in D(P)$, and the two contacts on the singleton give $n_a \geq 2$.

Conversely let $p = t^P$, $Y = u$. For $A = n_a = 2$ take $(x, Z) = (u^2, 3u)$. For even $A > 2$ take $(x, Z) = (u^A, u^{A-1})$ and for odd $A > 2$ take $(x, Z) = (u^{A-1}, u^A)$. Then $\min(v(x), v(Z)) = A-1$. For $k = n_b - 3$ choose $Q = 2$ if $k = 1$, $Q = 1 + u^{k-1}$ for odd $1 < k < 2P$, and $Q = 1$ if $k = 2P$. These are polynomials in t and the second terminal always contributes two to the quadratic fibre. This proves the classical equality since D_5 itself is allowed.

For E_6 the primitive A_3 term $s\xi$ leaves an A_3 with monodromy minus the transposition (1 4) on its four coordinates (Appendix B.3.4). In $u = s^2$ write them as $p + V, Y, Z, -p + V$, with p even, V, Y, Z odd, and $v(Y - Z) = 1$ for the internal A_1. Their weights are $(b, 0, 0, -b)$. The unique difference over $2b$ gives $\operatorname{ord}_{u^2} p = P$. By (6.13), $n_a = 2$ and

$$n_b = 2 + v(p + V - Y) + v(p + V - Z).$$

At least one summand after the two is one. The other belongs to $D(P)$. Conversely set $p = u^{2P}$, $V = 0$, $Y = u$. For $k = n_b - 3$ take $Z = 2u$ if $k = 1$, $Z = u^k$ if $1 < k < 2P$, and $Z = 0$ if $k = 2P$. Subtract the common mean, an odd polynomial, to impose trace zero.

For E_7 the primitive A_5 term leaves $A_2 + 3A_1$. The two simple forms of A_2 over b are exchanged, and their sum projects to $2b$. In $u = s^3$ their evaluations are $p + O, p - O$, with p even and O odd. Consequently $n_b = 3 + v(p + O)$ and $\operatorname{ord}_{u^2} p = P$. Take $p = u^{2P}$ and the odd witness for $k = n_b - 3$ above Lemma 6.10. The three orthogonal forms over a are cycled by c. Their common order is $2j$ for some $j \geq 1$, so $n_a = 2 + j$. Prescribe on them the three cyclic translates of s^{2j} independently of the A_2 forms. In both ambient types the polynomial in the surviving span starts after $s\xi$, and the mixed entries equal one by (6.13). □

For example, a D_6 presentation with a three-coordinate block has $n_{2b} = 1$: the leading terms of its three coordinates are $Us^2, U\omega s^2, U\omega^2 s^2$ on $t = s^6$, where $U \neq 0$ and ω is a primitive cube root. Each pair sum has order two. The D_5 row also allows $n_{2b} = 2$.

6.3.5 Two doubled rays and a tripled ray

Proof of the eight-label row of Theorem 6.13. The internal systems are $A_3 + 2A_1$ and $D_5 + A_1$, respectively. After the primitive term of the largest component the relevant span is D_4 with roots $\pm e_i \pm e_j$. In E_7 there is also an orthogonal internal A_1. In both models c exchanges e_1, e_2, fixes e_3, and negates e_4. Their projections are $(a, a, 2b, 0)$ and the internal root in D_4 is $e_1 - e_2$. The half-sum and half-difference of the roots over $2b$ give e_3, e_4, as in Appendix B.3.4.

In $u = s^{m/2}$ the four coordinate evaluations are $p + uQ, p - uQ, r, uH$, with $t = u^2$ and $Q(0) \neq 0$ by smoothness. The root over $2a$ gives $P = \operatorname{ord}_t p$. The four roots $e_i \pm e_4$ for $i = 1, 2$ have orders d_+, d_- twice each in u, so their contribution over a is $d_+ + d_-$. The two roots $e_3 \pm e_4$ have common order v_0 over $2b$. Each pair $e_i \pm e_3$ over $a \pm 2b$ has order one since r, p are even and uQ starts in degree one. Adding the backgrounds in (6.13) gives exactly $(A_0, B) = (2 + d_+ + d_-, 1 + v_0)$ of Lemma 6.10. Insert its polynomials after the primitive term $s\xi$. For the detached A_1 in E_7, add a nonzero degree-$m/2$ coefficient in that factor. The intersection $A_0 = 4$, $B = 2$ occurs for every P, including $P = 1$. □

Proof of the nine-label row of Theorem 6.13. Use the index-six extension of $Q(A_5) \oplus Q(A_2) \oplus Q(A_1)$ by $(\omega_1, -\omega_1, \omega_1)$. This vector has norm two, and the even extension has determinant $6 \cdot 3 \cdot 2/6^2 = 1$. In coset j the norm-two weights are the minuscule products with indices $(j, -j \bmod 3, j \bmod 2)$ for $1 \leq j \leq 5$. Together with the factor roots these are the 240 roots of E_8. Take A_5 and the difference of the first two A_2 weights internal. The three weights project to $w, w, -2w$, and the last factor's weights to $\pm q$. The roots project to precisely $\mathcal{B}_9$ and zero. The kernel lattice is the indicated internal root lattice. In fact an element in the internal real span has even glue index by its last component, and index divisible by three since its A_2 component lies on the internal root line. On that line pairing with an adjacent coroot makes the root coefficient integral. The labels $q + w, q + 2w$ generate the full quotient lattice. For each positive chamber a generic internal perturbation lifts it to an incident E_8 chamber, making these internal roots parabolic. The ordered table of §6.3.1 identifies the resulting presentation in each chamber.

Put $t = s^6$. The three factors have coordinates

$$x_j = f(\zeta_6^j s)\ (0 \leq j < 6), \qquad (X + U, X - U, -2X), \qquad \pm F,$$

where $\sum x_j = 0$, $f'(0) \neq 0$, $X, F \in t\mathbb{C}[[t]]$, $U(\zeta_6 s) = -U(s)$ and $\operatorname{ord}_s U = 3$. A sum of two distinct sixth roots of unity vanishes just for opposite pairs. For three it vanishes just for alternating triples, since three unit vectors summing to zero form an equilateral triangle. Put $R_i = x_i + x_{i+3}$ $(0 \leq i \leq 2)$ and $T = x_0 + x_2 + x_4$. From the two-subset and three-subset weights the remaining evaluations in (6.13), up to signs, are

w	$2w$	$3w$	q	$2q$
$R_i + X \pm U$	$R_i - 2X$	$3X \pm U$	$F \pm T$	$2F.$

For example the two-subset orbit has fifteen choices, of which twelve exit at degree one. The two A_2 weights over w give 24 exits and six remaining forms, hence background $24/6 = 4$.

By Lemma 6.11 with $r = 2, 3$, the common order h of the R_i is even and prime to three if finite, whereas T has degrees $3 \bmod 6$. Thus $R_i - 2X$ has order $2M$, where $M = \min(h/2, 3 \operatorname{ord}_t X)$ is finite since the fibre over $2w$ has finite orders. The six forms over w have

common order two if $M = 1$ and three otherwise. The two forms over $3w$ have order three. With $b = \operatorname{ord}_t F$, the fibre sums become

$$n_{2w} = 2 + M, \quad n_w = 4 + \min(2M, 3), \quad n_{3w} = 1, \qquad n_q = 3 + \min(2b, \operatorname{ord}_s T/3), \quad n_{2q} = b.$$

The two mixed pairs have already exited. This proves necessity.

For any prescribed $M = n_{2w} - 2$ and $k = n_q - 3 \in D(b)$ choose

$$(R, X) = \begin{cases} (s^{2M}, 0) & 3 \nmid M, \\ (0, t^{M/3}) & 3 \mid M, \end{cases} \qquad T = \begin{cases} s^{3k} & k < 2b, \\ 0 & k = 2b. \end{cases}$$

Take $f = s + R/2 + T/3$, $U = s^3$ and $F = t^b$. The opposite-pair sum at $i = 0$ is exactly R, and the alternating-triple sum is exactly T. Their other translates follow by equivariance. The degrees in f avoid multiples of six, so its trace vanishes. Its degree-one term smooths A_5 and U smooths the internal A_1. This one polynomial arc realises the two radial choices simultaneously. □

6.3.6 Long-root contacts in the two doubled-short phases

Proof of the doubled-short row of Theorem 6.13. Both covers have degree six. In E_7 the internal system is $A_2 + 3A_1$. The primitive A_2 coefficient at degree two leaves A_5. Each s has twelve degree-two exits and each $2s$ has three. Hence their values are 4 and 1. The remaining internal reflections transpose the members of three pairs in A_5. Each long label has the four differences between one pair of blocks. In $u = s^3$ these are smooth quadratic blocks. Lemma 6.12 gives $n_\ell = 1 + h_\ell$ with the stated triangle condition, and constructs the trace-zero polynomial tail. Adding it after the degree-two coefficient smooths all three A_1's.

In E_8 the internal system is $D_4 + A_2$. The primitive D_4 coefficient at degree one leaves E_6, and the A_2 coefficient at degree two leaves $A_2 \oplus A_2$. At these degrees the respective character spaces are lines, and smoothness makes both coefficients nonzero. The numbers of roots exiting at degrees $(1, 2)$ per label in $S, 2S, L$ are respectively $(18, 6), (0, 3), (6, 0)$ (Appendix B.3.4). Thus $n_s = (18 + 2 \cdot 6)/6 = 5$, $n_{2s} = 1$, and each long background is one. The two remaining roots over each long label are corresponding roots of the exchanged A_2 factors. The two simple-root pairs project to $(1, 0)$ and $(1, 3)$, and c^2 fixes them pointwise. Thus in $u = s^3$ every projected arc is an ordinary arc in one factor and its translate under $u \mapsto -u$ in the other, using the identification induced by c. A difference of order h contributes $2(3h)/6 = h$. The minimum of the three difference orders occurs at least twice.

Explicitly relabel the three scalar indices so that $h_{12} = p \geq h_{13} = h_{23} = q$, with $h = n_L - 1$. The polynomials $0, u^p, 2u^q$ have these differences, also when $p = q$. Subtract their mean for A_2. The construction in the preceding paragraph adds this pair of arcs after the two primitive coefficients, so it preserves all internal orders. In E_7 the same three polynomials with the common linear shift of Lemma 6.12 give the other phase. The table of marked presentations exhausts both possibilities, and the short values separate their images at every long triple. □

A Low-rank exceptional supports and numerical conditions

Tables A.1–A.3 describe the 27 additional signed supports in their actual curve lattices. Each representative lies in the displayed $\mathbb{Z}^r$. A string of r digits is one vector, so that, for instance, `021` denotes $(0, 2, 1)$, with its multiplicity as a lattice vector intact. In the tables $k \mid L$ assigns the value k to each vector in the list L. Adjoin both signs of every listed vector, including the vectors in the right column. These two columns specify the *entire signed support*. In that row let $t_1, \ldots, t_m$ denote the vectors in the right column in the printed order and put $x_i = n(t_i)$; every x_i is a positive integer. Section 4.3 gives the marking convention. Section 4.3.1 works through row 2e.

Numerical notation. Write

$$D(a,b): \quad b \geq 1, \qquad a \in \{\, 2j : 2 \leq j \leq b+1 \,\} \cup \{2b+3\},$$
$$T(a,b): \quad (a,b) = (6,3) \quad \text{or} \quad a = 7,\ b \geq 4.$$

Write $M(a,b,c)$ when the minimum occurs at least twice.

A.1 Rank two

Table A.1: Complete support and input coordinates in rank 2.

Row	$\lvert B\rvert$	Fixed values $k \mid L$	Other vectors $t_1, \ldots, t_m$
2a	6	1 | 02 10 13 2 | 11 12 4 | 01	—
2b	7	1 | 02 10 13 3 | 11 12 5 | 01	23
2c	8	1 | 01 30 41 2 | 11 31	10 20 21
2d	10	1 | 01 30 51 2 | 11 41 3 | 20 21 31 6 | 10	52
2e	10	1 | 10 14 23 25	01 02 11 12 13 24
2f	11	1 | 10 15 24 26 2 | 02 11 14 4 | 01 12 13	25
2g	12	1 | 01 40 61 2 | 11 30 51 3 | 21 41 5 | 20 6 | 10	31 62
2h	14	1 | 22 30 62 63 2 | 01 32 41 52 4 | 11 31	10 20 21 42

Conditions on the variable labels.

2a All entries are fixed.

2b, 2d The listed positive variable is free.

2c $T(x_1, x_2)$, $x_3 \geq 3$, and $\min(x_2, x_3) = 3$.

2e Put $(P, A, R, Q, S, D') = (x_1, x_2, x_3, x_4, x_5, x_6)$. Require $A, D' \geq 2$ and put

$$L(b) = \{6\} \cup \{2j : 4 \leq j \leq b+1\} \cup \{2b+3\}.$$

Allow the union of the following alternatives:

$$\begin{array}{ll} Q = 6,\ R = S = 4, & P \in L(A), \\ P = 6,\ R = S = 4, & Q \in L(D'), \\ P = Q = 6,\ R, S \geq 5, & \text{the mixed test below holds.} \end{array}$$

For the mixed test set $(a, d, u, v) = (2A - 2, 2D' - 2, R - 3, S - 3)$ and $h = \min(a, d, u, v)$. An even h must occur at least three times. An odd h must satisfy $u = v = h < \min(a, d)$.

2f $x_1 \geq 2$.

2g $D(x_1, x_2)$.

2h $T(x_1, x_2)$, $T(x_3, x_4)$, and $\min(x_1, x_3) = 6$.

A.2 Rank three

Ordinary and folded root tests in ranks three and four. For a set I of column indices let $R_I = \{\pm t_i : i \in I\}$. The notation $*$ denotes all indices in the current row. The ordinary test $O(I)$ requires each vector of depth x_i to lie outside the $\mathbb{Q}$-span of the vectors of strictly greater depth. For a reduced core R_I of type B_q, C_q or G_2, write $F(I)$ for the following test. Distinguish its short and long roots using the invariant form of the core in its own span. Put $e = 2, 2, 3$, respectively, and form

$$Q_I = \mathbb{Z}R_I, \qquad D_I = Q_I/\mathbb{Z}R_{I,\text{long}}, \qquad d_s = n_s - (e - 1), \qquad d_\ell = en_\ell.$$

Require $d > 0$. At each attained k, each exact-depth row must be outside the span of all strictly deeper rows. For $e \mid k$ use the label vectors over $\mathbb{Q}$. Otherwise use their images in D_I over $\mathbb{F}_e$. In particular one takes the root lattice of the *core*, even when it has finite index in its ambient saturation. For such a core $s(I)$ denotes the minimum of its short-root GV values. The finite-field test describes dependence relations of characteristic-zero coefficient forms. The construction of an arc uses the latter coefficient space over an infinite field.

Table A.2: Complete support and input coordinates in rank 3.

Row	$\lvert B\rvert$	Fixed values $k \mid L$	Other vectors $t_1, \ldots, t_m$
3a	11	1 | 001 021 100 111 120 131	010 011 020 110 121
3b	12	1 | 001 010 110 201 212 312 2 | 100 211	101 111 202 311
3c	13	1 | 010 012 110 112 121 123	001 011 100 111 122 133 233
3d	14	1 | 001 021 022 100 110 133 143 2 | 111 121 122 132 4 | 011	010 243
3e	14	1 | 001 011 111 131 231 241	010 100 110 120 121 130 230 242
3f	16	1 | 100 120 122 142	001 010 011 020 021 022 110 111 121 131 132 242
3g	17	1 | 124 134 135 234 235 245 246 2 | 001 011 012 111 112 122 4 | 123	010 100 110
3h	18	1 | 001 010 110 202 212 302 311 413 513 3 | 100 201 211 312 5 | 101	111 301 412 523

Row	$\lvert B\rvert$	Fixed values $k \mid L$	Other vectors $t_1, \dots, t_m$
3i	20	1 | 001 201 221 311 321 423 523 533 2 | 100 110 312 322 4 | 211	010 101 111 202 212 222 422
3j	21	1 | 010 011 221 300 410 411 621 2 | 110 111 310 311 321 521	001 100 200 210 211 421 631 632
3k	23	1 | 022 100 120 133 153 242 243 263 264 2 | 001 021 111 122 131 142 4 | 011 121 132	010 020 110 143 253

Conditions on the variable labels.

3a $D(x_1, x_3)$. The triple (x_2, x_4, x_5) has minimum $m \geq 2$ attained at least twice. Require $x_1 = 4$ or $m = 2$.

3b $D(x_1, x_3)$, $x_2 \geq 2$, and $x_1 = 4$ or $x_2 = 2$. The fourth coordinate is free.

3c $F(I)$ for $I = * \setminus \{1\}$ (type G_2). Require $x_1 \geq 3$, and either $x_1 = 3$ or every short value is 3.

3d $x_1 \geq 2$ and $x_2 \geq 1$ independently.

3e $D(x_5, x_8)$ and $F(* \setminus \{5, 8\})$ (type G_2).

3f The variable labels form full BC_3; apply Theorem 3.3 in the lattice generated by the primitive vectors of the three doubled rays. Equivalently use the class factorisation of Theorem 3.1; this is the form used in the integer comparison of Appendix B.3.1.

3g $M(x_1, x_2, x_3)$.

3h $F(*)$ (type B_2).

3i Take $u = 101$, $v = 111$, $w = 211$. On the ordered *test* vectors $(u, 2u, v, 2v, w, u + v, u - v)$ assign depths

$$(x_2 - 3,\ 2x_4,\ x_3 - 3,\ 2x_6,\ x_7 - 1,\ x_5 - 1,\ x_1 - 1).$$

Require positive depths and exact-depth noncontainment among deeper test vectors over $\mathbb{Q}$ at even levels and in $\mathbb{Z}^3/2\mathbb{Z}^3$ at odd levels. In addition $x_7 > 2$ implies $x_2 = x_3 = 4$. In particular the test vector w reads the coefficient at $2w$.

3j $T(x_2, x_3)$ and $F(I)$ for $I = * \setminus \{2, 3\}$ (type G_2). If $x_3 > 3$, every short value of R_I is 3.

3k $D(x_1, x_2)$. The triple (x_3, x_4, x_5) has minimum $m \geq 2$ attained at least twice. Require $x_1 = 4$ or $m = 2$.

A.3 Rank four

Table A.3: Complete support and input coordinates in rank 4.

Row	$\lvert B\rvert$	Fixed values $k \mid L$	Other vectors $t_1, \dots, t_m$
4a	17	1 | 0001 0010 0110 0121 1110 1121 1221 1232	0011 0100 0111 0122 1000 1100 1111 1122 1222
4b	18	1 | 0001 1000 1100 1110 2101 2111 2211 3212	0010 0100 0110 1001 1101 1111 2102 2112 2212 3211

Row	$\lvert B\rvert$	Fixed values $k \mid L$	Other vectors $t_1, \ldots, t_m$
4c	21	1 | 0001 0010 0110 0121 0122 1110 1121 1122 1221 1222 1233 2 | 0011 0111 1111 1232	0100 1000 1100 1243 1343 2343
4d	23	1 | 0001 0011 0211 0221 1000 1100 1110 1210 1222 1322 1332 1432	0010 0100 0110 0111 0210 0222 1111 1211 1221 1321 2432
4e	29	1 | 0001 0100 0110 1100 1110 2001 2102 2112 2211 3102 3112 3211 4213 5213 2 | 1000 2101 2111 3212	0010 1001 1101 1111 2002 2212 3101 3111 4212 5313 5323
4f	30	1 | 0001 0010 0100 0110 0121 0122 1100 1110 1121 1122 1211 1233 1332 1343 2332 2343 2443 2454	0011 0111 1000 1111 1210 1221 1222 1232 1243 1333 2333 2453
4g	33	1 | 0122 1122 1222 1233 1242 1243 1342 1343 1353 2342 2343 2353 2453 2463 2464 2 | 0001 0011 0111 0121 1111 1121 1221 1231 4 | 1232	0010 0100 0110 0120 1000 1100 1110 1120 1220
4h	36	1 | 0010 0011 0210 0211 1000 1100 1220 1222 1320 1322 1442 1542 2431 2432 2652 2653	0001 0100 0110 0111 0220 0221 0222 1110 1111 1210 1211 1221 1321 1331 1332 1431 1432 2442 2542 2642

Conditions on the variable labels.

4a, 4g $F(*)$ (type B_3).

4b $F(* \setminus \{10\})$ (type B_3); x_{10} is free.

4c $O(*)$ (type A_3).

4d $D(x_4, x_6)$ and $F(I)$ for $I = * \setminus \{4, 6\}$ (type C_3). Require $x_4 = 4$ or $s(I) = 2$.

4e $D(x_2, x_5)$ and $F(I)$ for $I = * \setminus \{2, 5\}$ (type B_3). Require $x_2 = 4$ or every short value of R_I is 2.

4f Both $F(I)$ and $F(J)$, where $I = \{1, 5, 6, 8, 9, 12\}$ and $J = \{2, 3, 4, 7, 10, 11\}$ are G_2 cores. At least one core must have all three short values equal to 3.

4h Apply the following capacity test to the variable BC_4 support.

For 4h the four primitive doubled directions u_i form a basis of the model lattice. Put $a_i = n(u_i)$, $b_i = n(2u_i)$, $p_{ij} = n(u_i - u_j)$, $q_{ij} = n(u_i + u_j)$. Require $a_i \geq 4$, $b_i \geq 1$, $p_{ij}, q_{ij} \geq 2$ and $p_{ij} > 2 \iff q_{ij} > 2$. Adjoin the diagonal to $p_{ij} > 2$ and require an equivalence relation. Each class has either all $a_i = 4$ or all $a_i > 4$; call the latter active. The possible active classes are as follows:

class sizes	allowed activity
4	none
$3 + 1$	none or the triple
$2 + 2$	none or one double
$2 + 1 + 1$	the double is optional; at most one singleton
$1 + 1 + 1 + 1$	at most two singletons.

In each class use $2u_i$ of depth $2b_i$ and $u_i \pm u_j$ of depth $p_{ij} - 1, q_{ij} - 1$, respectively (minus takes p, plus takes q). Add u_i of depth $a_i - 3$ in each active class. At each attained depth an exiting vector must avoid the span of the deeper vectors in its class, over $\mathbb{Q}$ in even degree and in the actual $\mathbb{Z}^4/2\mathbb{Z}^4$ in odd degree.

B Polynomial witnesses and finite verification certificates

This appendix collects the explicit low-rank polynomial witnesses (Appendix B.1), the integral support inventory and chamber maps (Appendix B.2), and the counts for the integer comparisons, reduced folds, unit fibres, and instructions for reproducing the calculations (Appendix B.3).

B.1 Polynomial witnesses for the low-rank tables

For row 2e one can check the coefficient plane of Figure 1 directly. Number the simple roots of E_8 along the chain 0 to 6 with vertex 7 adjacent to 2. Take $J = \{0, 3\}$ and the internal Coxeter product $r_1r_2r_7r_4r_5r_6$, where r_j denotes simple reflection. In simple-root coordinates choose the degree-one coefficient $(0, i, 1+i, 0, -1, -1+i, i, 1)$, with $i^2 = -1$. Every internal root has order one. At degree two write the coefficient as $x(\alpha_4 + \alpha_6) + y(\alpha_1 + \alpha_7)$. Representatives over $2w, 2v, v - w, v + w$, respectively, have simple-root coordinates

$$(0, 1, 2, 2, 1, 1, 0, 1), \quad (2, 3, 5, 4, 3, 2, 1, 2), \quad (1, 1, 1, 1, 0, 0, 0, 0), \quad (1, 2, 3, 3, 2, 1, 1, 1).$$

Pairing each with the degree-two coefficient using the Cartan matrix gives the four forms whose kernels appear in Figure 1. At odd degrees the first two restrict to zero and the last two are proportional and nonzero. For example, the degree-three vector $(0, -i, 1 - i, 0, 0, 0, 0, 1)$ evaluates on these four representatives as $(0, 0, -1, -1)$. The degree-four vector $(-8, -4, 0, 4, 3, 2, 1, 0)$ evaluates to 10 and -4 on the two doubled labels. Together with the first coefficient above these two vectors give the explicit arc for $(A, R, S, D') = (3, 6, 6, 3)$: put them at degrees three and four and set the degree-two coefficient to zero.

Choices in the polynomial witnesses. In row 2f the surviving pair at matched amplitude has common character zero when its required order is an even multiple of five. Its projected label in the printed basis is $(2, 5)$. Choose the invariant coefficient η with $(2, 5)(\eta) \neq 0$; the first dual basis vector has evaluation two. A nonzero invariant vector can lie in the kernel of this label. The nontrivial odd character supplies the alternative orders.

For row 2g put $v = \texttt{31}$. The Coxeter product interchanges the two matched roots. Writing their sum as θ, their mean is invariant and their difference is anti-invariant in $z = s^6$. If $n(2v) = k$, the invariant term with $\theta(\eta) = 2$ must be present at degree $12k$, including on the unmatched $n(v) = 4$ branch. In a matched branch a finite odd correction gives the two evaluations $s^{12k} \pm s^{6d}$, where d is odd and at least three. Their common order in z is $\min(2k, d)$. Omitting the odd correction gives order $2k$.

For row 3b write $u = \texttt{101}$, $p = \texttt{111}$ and $q = \texttt{311}$. In the usual even/odd scalar model let $t = z^2$, $k = n(2u)$, $d = n(p) - 1$, and take the block mean $a = t^k$, $U = z$ and $Q = t^{n(q)}$. On the baseline $n(u) = 4$ take $(P, W) = (t^{d/2}, 0)$ for even d and $(P, W) = (0, 2z^d)$ for odd d. Both $U \pm W$ then have order one, including at $d = 1$, and the doubled and independent line labels have their specified orders.

The single-line and four-block lattices. For row 3c one may use the ternary glue code generated by $(1, 1, 1, 0), (1, -1, 0, 1)$ in the discriminant group of A_2^4. Take the first two factors to be internal and intersect with the perpendicular of one root in the fourth factor. With the same standard-weight convention in the first two factors, the core short fibres from the first glue word see $U\omega^i + V\omega^j$; the line fibre from the second sees $U\omega^i - V\omega^j$, where $UV \neq 0$ and $\omega^3 = 1$ is primitive. The respective matches require $U/V \in -\mu_3$ and $U/V \in \mu_3$. Reversing the convention in one factor interchanges both signs. The resulting disjunction is the one printed in row 3c.

In the baseline construction for 4e, write the two odd series as $U = z$ and W. If W has order one, choose its final leading coefficient in $\mathbb{C} \setminus \{0, 1, -1\}$. This keeps every folded B_3 short observation, since those observations see only the order of W. Both $U + W$ and $U - W$ now have order one, as required for the radial baseline. If W has higher order the same equalities of orders hold with no adjustment.

For row 4h take the standard E_8 lattice and the internal roots $\alpha_i = e_{2i-1} - \epsilon_i e_{2i}$, where $\epsilon = (1, 1, 1, -1)$. The projection has coordinates $\beta_i = y_{2i-1} + \epsilon_i y_{2i}$, which are integral on the whole lattice and attain the standard basis. If $\beta = 0$ then the first three coordinate pairs sum to zero and the last two coordinates are equal. A half-integral lattice vector in this kernel would have odd total coordinate sum, in contradiction with the even-sum condition on the half-integral coset of E_8. The integral kernel consists exactly of the integral combinations of the α_i. This proves the asserted lattice and justifies taking residues in $\mathbb{Z}^4/2\mathbb{Z}^4$. Smoothness makes the four odd leading amplitudes nonzero. Equal squares partition the indices; an active primitive label requires a signed sum of the other three amplitudes to vanish. The possible simultaneous matches give the capacity table. The finite comparison below checks both these first-order restrictions and every higher coefficient.

B.2 The exceptional support count and its chamber maps

The following counts use the positive chamber marking of Section 4.2. A marked type is counted once in its rank, even when several parabolic presentations give the same positive support.

curve rank	1	2	3	4	5	6	7	8	total
exceptional support types	6	38	94	113	81	36	9	1	378
families in the first alternative	6	13	11	10	81	36	9	1	167
families in the second alternative	0	25	83	103	0	0	0	0	211.

The finite support calculation. Generate all roots of E_6, E_7, E_8 from the simple roots by closing under simple reflections, and project to the simple-root coordinates indexed by each nonempty subset J. This gives $63 + 127 + 255 = 445$ presentations and 378 positive supports modulo basis permutation. The two integral comparisons of Section 4.2 give the following disjoint inventory.

curve rank	2	3	4	total
additional signed configurations	8	11	8	27
additional marked positive types	25	83	103	211
exceptional presentations of these types	35	86	107	228

This calculation compares full vectors and their lattices. In a fixed rank a putative lattice isomorphism is determined by the images of a basis. The form M_B^{-1} can be used to discard impossible images before checking the determinant and the full signed set. The classical reduction gives precisely the classical supports in the inventory of Section 4.2; none of these 27 configurations is classical. For instance a reduced rank-four classical support has at most sixteen sign-pairs: a nonterminal D frame has twelve diagonal directions and at most four axes, a reduced terminal frame has at most sixteen directions, and A_4 has ten. The bound excludes the reduced eighteen-label row.

How to check the exceptional inventory. The following root formula determines all representatives and all integral changes of basis. Number the central node 3 and the legs $(2, 1), (4, \ldots, n-1), (n)$. If its coefficient k is zero, take an interval on one leg. Otherwise add a terminal zero to each leg. On a leg of edge length h write $k = hq + s$ and take s steps of size

$q+1$ and $h-s$ steps of size q, starting at k. Every ordering of the steps is allowed. Indeed the squared root length equals the sum of all squared steps minus k^2. At fixed sum the minimum occurs when the steps differ by at most one. For E_6, E_7, E_8 this integer minimum is two for $1 \le k \le 3, 4, 6$ respectively. Cauchy–Schwarz excludes larger levels except $k = 7$ in E_8, whose minimum is four. The numbers of roots at the successive levels starting at zero are $(7, 18, 9, 2)$, $(10, 24, 18, 8, 3)$ and $(14, 30, 30, 20, 15, 6, 5)$. Deleting the coordinates outside J now gives each support and each root fibre.

For the positive supports use $s_i = \sum_{\beta \in B} \beta_i$ and $t_i = \sum_{\beta \in B} |\beta|_1 \beta_i$. The cardinality together with the sorted pairs (s_i, t_i) distinguishes all 378 representatives. For every repetition test the coordinate permutations preserving these pairs and compare all vectors. In ranks two to four the 42 signed families have distinct triples $(r, |B|, \det M_B)$. Within each family verify the entire signed image for a unimodular basis map and apply the following chamber moves. For $j \in J$ apply the longest element on $I \cup \{j\}$, with I the complement of J. It replaces the marked node j by its opposite in its connected component and maps the internal simple roots to negatives of the new internal simple roots. Conjugating within the new internal Weyl group aligns the Coxeter elements without changing any quotient label. Thus the move transports all orders in the complete fibres and the internal smoothness condition. Applying these moves gives a path to a representative of its ambient orbit for each of the $50+91+154 = 295$ low-rank exceptional presentations.

B.3 Comparing the table with all root orders

The support comparison gives the 228 compatible exceptional presentations. To verify a numerical row we also compare its image with the integer spectrum of each such presentation. This comparison treats the multiplicities as unbounded variables.

B.3.1 The integer comparisons by curve rank

For each row, Section 2.3.3 gives the integer image of each compatible presentation. The counts for Proposition 4.3 are

r	rows	marked types	presentations	initial patterns
2	8	25	35	59
3	11	83	86	370
4	8	103	107	729
	27	211	228	1 158.

(B.1)

An initial pattern is counted up to independent rotations of the internal Coxeter components. For each pattern the calculation checks the forward implication in (4.2). It checks the reverse implication for 226 individual presentations and for all 27 unions, and checks invariance on 66 generators. These 1 477 queries, together with the two exclusions in row 3f, give 1 479 unsatisfiable counterexample or exclusion queries. The 30 satisfiable queries consist of one nonemptiness check per union, membership of the example in the printed row, and its two E_7 realisations. Section 4.5 gives a direct calculation for that example.

The row 3f obstruction in the numbered basis. For the E_8 presentation with one-based $J = 157$, the internal Coxeter product in increasing node order has no degree-two character. The degree-three column is $(0, x+z, 0, x, 0, y, 0, z)$ with $y \neq 0$ for the internal A_1 at node 6. To test degree-one vanishing, pair the roots with the two rational columns $(0, 1, 1, 0, 0, 0, 0, 0)$ and $(0, 1, 0, -1, 0, 0, 0, -1)$ spanning $\ker \Phi_6(c)$. Over label 010, twelve roots exit in degree one and the remaining four have degree-three kernels $x + 2z \pm y = 0$. Over 011, six roots exit in degree one and the other two have kernel $x + 2z = 0$. If all the remaining roots exit in degree

three, the two values are 4 and 2. The values 5 and 3 of (4.3) therefore require both a zero of $x + 2z \pm y$ and a zero of $x + 2z$. This forces $y = 0$. The same obstruction is expressed in Euclidean coordinates in Section 4.5.

B.3.2 The exceptional folds of reduced root supports

Consider the exceptional presentations of a reduced G_2 support. On $t = s^m$ their first admissible coefficient has degree one. The following entries count, over each external label, the roots exiting at degree one and those left in a fixed D_4 subsystem:

	m	short label	long label
$E_6/(A_2 + A_2)$	3	$(6, 3)$	$(0, 1)$
E_7/A_5	6	$(12, 3)$	$(0, 1)$
E_8/E_6	12	$(24, 3)$	$(0, 1)$

In the first line the counts use equal primitive amplitudes. A short fibre there has first evaluations proportional to $U\omega^i - V\omega^j$ for $i, j = 0, 1, 2$, with $\omega^3 = 1$ primitive. Independent internal Coxeter rotations take each cancellation to $U = V$. If none occurs, all short roots already exit at degree one. The other two internal systems have a single primitive eigenline. Compute its annihilating roots by restriction to the primitive character and take the indecomposable positive roots in that subsystem. There are four in each model. Their pairings give the D_4 diagram, with its central root fixed by c and the three leaves permuted cyclically.

For F_4, the annihilator is E_6 with its diagram involution. In $E_7/3A_1$ the short and long counts are $(2, 2), (0, 1)$ on the double cover. In E_8/D_4 they are $(6, 2), (0, 1)$ on the sixfold cover. In the first model each short fibre sees four signs $\pm U_i \pm U_j$. By changing the signs of the three internal amplitudes, arrange that those with equal squares agree. Its first zero set is then contained in the one with all amplitudes equal. The second model has a single primitive eigenline.

Write $e = 3$ for triality and $e = 2$ for the involution. On the invariant character compare orbit sums with the marked label vectors. On each moving character compare the spans of the orbit differences with the residue spans in the quotient of the root lattice by the long-root lattice, as in (2.4). In triality the two moving eigenspaces are conjugate lines. Orthogonal projection of the arc to the displayed subsystem preserves the evaluations of its roots. On its span c has order e, so the projected arc is a series in $u = s^{m/e}$, equivariant for the diagram action. If a short orbit exits in u-degree d, its e roots contribute d to the normalised fibre sum. The exterior roots contribute $e - 1$ by the table. A long label has one root and contributes d/e. These are exactly the depths in (2.4). Conversely take a polynomial folded arc $\eta(u)$ passing the character tests. Add Msv to $\eta(s^{m/e})$, where v is the displayed matched primitive vector. On the subsystem this addition has zero evaluation. Outside it every first evaluation is nonzero for all but finitely many M, including those on the internal roots. A single such choice gives smoothness and all the prescribed orders, including depth one.

B.3.3 Unit fibres and classical overlaps

For each external label let a be the unique internal-dominant root in its fibre. Write $p_i = 2a_i - \sum_{j \sim i} a_j$ at the internal nodes. The condition of Lemma 5.2 holds precisely when the tuple $(p_i)_{i \in I}$ is a single 1 at an endpoint of a path component and zero elsewhere. Indeed these two endpoint weights are exactly the standard and dual-standard fundamental weights of that component; an isolated vertex also counts as an endpoint. The other factors act trivially. For each pair of external label and endpoint, solve $C_{II} a_I = e_j - C_{IJ}\beta$, where C is the Cartan matrix, and test for squared length two. Exhaustion follows by applying the endpoint test to the explicit positive roots above. Each chamber move preserves the number of these fibres. In

particular the two rank-two families with only two such fibres have six presentations each. All the other forty nonfull rank-two presentations have at least three. This proves the exceptional count in Lemma 5.3. For the nonfull presentations other than the spin extension, the number of presentations and the minimum count in each rank are

curve rank	presentations	minimum certified unit fibres
1	10	1
2	52	2
3	106	4
4	118	8
5	83	10
6	35	16
7	8	28.

The complete histogram of certified unit fibres is

r	number of unit fibres : number of presentations
2	$2:12,\ 3:32,\ 4:8$
3	$4:24,\ 6:22,\ 7:32,\ 8:10,\ 9:18$
4	$8:16,\ 9:5,\ 10:6,\ 11:18,\ 12:10,\ 14:20,\ 15:28,\ 16:7,\ 18:8$
5	$10:6,\ 15:6,\ 16:22,\ 24:21,\ 25:28$
6	$16:7,\ 27:7,\ 32:21$
7	$28:8.$

The four rank-three spin rows, with zero certified unit fibres, are omitted from this second table. Each counted fibre contributes one to $U(n)$.

The following exceptional presentations occur in Section 2.2.6 and in the classical frame reduction of Section 3.2. Exceptional Dynkin diagrams are numbered along the chain from zero, with the last vertex adjacent to vertex two.

support	J in the exceptional diagram	comparison
A_2	$E_6 : J = 04$	reduced rank-two overlaps
B_2	$E_6 : J = 01, 05, 34, 45;\ E_7 : J = 05, 45$	reduced rank-two overlaps
C_3	$E_7 : J = 045$	classical exceptional folds
$N_3(\{2,3\})$	$E_6 : J = 014, 034$	partial classical folds
$N_3(\{1,3\})$	$E_6 : J = 015, 345$	partial classical folds
$N_3(\{1,2\})$	$E_6 : J = 045$	partial classical folds
F_4	$E_7 : J = 0124;\ E_8 : J = 0456$	folded-tail comparisons
G_2	$E_6 : J = 25;\ E_7 : J = 01;\ E_8 : J = 56$	folded-tail comparisons
BC_2	$E_7 : J = 04;\ E_8 : J = 06$	two-anchor absorption.

The rank-three classical overlaps. Use the one-based chain numbering with node n attached to node 3. Take first E_6 with $J = 125$ and $t = s^4$. Its internal system is A_3. With $i^2 = -1$ a primitive vector in simple-root coordinates is $(0, 0, 1+i, 1, 0, 1)$. The simple positive roots annihilating it are `011001`, `011100`, `100000`. The third meets the first two and they are orthogonal to each other. The Coxeter product exchanges the first two and fixes the third. The list of roots in Appendix B.2 gives the following counts on each of the indicated fibres:

	labels	degree-one exits	surviving roots
$E_6/125$	$001, 011, 111, 121$	4	0
	$010, 110$	4	2
	$100, 120$	0	1
$E_7/156$	$010, 011, 100, 110, 111, 121$	6	2
	$001, 021, 221$	0	1

In the first model write $2u = \texttt{100}, 2v = \texttt{120}$ and $h = (1/2, 1, 1)$ in the rational coordinate frame. Its four mixed labels $h \pm u, h \pm v$ have value one. The two short labels are $v - u, u + v$ and the two long labels are $2u, 2v$. The involution on the surviving A_3 gives the C_2 test on these four labels. For the second model take E_7 with $J = 156$ and $t = s^6$. For $z = \exp(2\pi i/6)$ use $(0, z, 1 + z, 1, 0, 0, 1)$. Its annihilator has simple roots, in path order,

$$\texttt{1110001},\ \texttt{0011100},\ \texttt{0000010},\ \texttt{0111101},\ \texttt{1111000}.$$

The Coxeter product reverses this A_5 path. Taking $2u = \texttt{001}, 2v = \texttt{021}, 2w = \texttt{221}$ identifies the six short labels with the pairwise sums and differences up to sign. This gives full C_3.

Indeed in the two models put $y = s^{m/2}$ for $m = 4, 6$, respectively. Projection to the surviving span gives an involutive arc in y. For a short label the two roots of order d in y contribute d, in addition to the background one. For a long label the single root contributes $d/2$. Conversely add a nonzero multiple of s times the primitive vector to any such folded polynomial arc in y. Every root outside the surviving subsystem, including every internal root, then exits in degree one. The listed coordinates prove the two image equalities. The chamber moves of Appendix B.2 carry $E_6/125$ to the other four E_6 presentations.

The rank-two classical overlaps. For A_2 the three fibres in $E_6/15$ each have six first exits and two survivors on $t = s^6$. The survivors form two A_2 factors exchanged by the Coxeter product. On $y = s^3$ the projected arc is specified by an ordinary arc in one factor, the other being its transport under $y \mapsto -y$. Thus the three values are $1 + d_\beta$ for an ordinary A_2 contact triple d. Their minimum occurs at least twice.

For B_2 each short fibre consists of ten first exits on the fivefold cover of $E_6/12$, or sixteen first exits on the eightfold cover of $E_7/16$. Both short values equal two. For any positive long values p, q, take singleton coordinates $(t^p + t^q)/2, (t^q - t^p)/2$ and terminal coordinates $s, 2s$ on $t = s^2$. These give the long values p, q and both short values two. The listed exceptional images are therefore contained in the classical image. For the two full-BC_2 presentations the surviving scalar weights are $(u_1, u_2, 0, 0, -u_2, -u_1)$ in $E_7/15$ and $(u_1, u_2, 0, -u_2, -u_1)$ in $E_8/17$. The two zero weights in E_7 differ by the internal A_1 root. Section 3.3.2 constructs the classical arc in each case.

The marked chambers of the F_4–line family. The rank-five F_4–line family consists of the 21 marked chambers of the signed configuration

$$\Phi_{F_4} \ \cup\ \{\pm v, \pm 2v\} \ \cup\ \{s + v, s - v : s \in \Phi_{F_4,\mathrm{short}}\}, \qquad \Lambda = Q(F_4) \oplus \mathbb{Z}v.$$

It has 100 signed labels and 50 positive labels. To see this directly, in the Euclidean E_8 model take internal roots $e_5 - e_6, e_5 + e_6, e_7 - e_8$ and project to the first four coordinates and $v = (e_7 + e_8)/2$. The integer roots give the D_4 long roots, the short coordinate vectors and their $\pm v$ shifts, and $\pm v, \pm 2v$. The half-integer roots give the other sixteen F_4 short roots and their $\pm v$ shifts. The projected lattice is exactly the one displayed. Theorem 6.7 supplies its complete numerical criterion, including the necessary coupling between its F_4 core and its doubled direction.

B.3.4 Reproducing the calculations

The code accompanying this paper is distributed as `Marked_GV_Verification_Code.zip`. Put the five Python files in one directory. With SymPy and `z3-solver` installed, run

```
python3 atlas.py
python3 finite.py
```

The first command checks every row in Appendix A as in Proposition 4.3. The second generates the exceptional supports and chamber maps, checks the marked root fibres, and compares 41 additional images over the positive integers. These are 17 presentations of the four rank-two families, four of the spin extension of B_3, and 20 in the reduced and classical overlaps. For B_2 and BC_2 the required inclusion is from the exceptional presentation into the classical image. For the other cases the comparison tests equality with the stated image. The program also evaluates the two coefficients of the polynomial in Section 4.5 on all 126 signed E_7 roots.

Both commands accept `--paper` followed by this paper's TeX file to read the fixed entries and ordered vector columns directly from Appendix A. The same columns and the indexed arithmetic rules are given in `rows.py`. With the TeX source, `atlas.py` also compares the computed tuple and rank totals with (4.4) and (B.1). For the worked example run `python3 atlas.py --row 3f`. Adding `--smt-dir queries` writes the individual comparisons as SMT-LIB files for an independent integer solver. Run with Python assertions enabled.

C Summary of notation and terminology

This appendix summarises the recurring symbols, numerical test operators, and support-family names used across the paper, together with their definitions and primary locations.

C.1 Core geometric and root-system notation

Symbol	Meaning	Location
$\Lambda, (c_1, \dots, c_r)$	Curve lattice $\bigoplus_{i=1}^{r} \mathbb{Z}c_i$ with ordered basis of irreducible exceptional curve classes	§1.1
$n,\ B,\ B^{\pm}$	Marked input $n : \Lambda_{\geq 0} \setminus \{0\} \to \mathbb{Z}_{\geq 0}$, positive support $B = \{\beta : n_\beta > 0\}$, and signed support $B^{\pm} = B \cup (-B)$	§1.1, §2.1
$M_B,\ (x, y)_B$	Positive moment matrix $\sum_{\beta \in B} \beta\beta^{\mathsf{T}}$ and coordinate-free inverse metric $x^{\mathsf{T}} M_B^{-1} y$	§1
$\Phi,\ I,\ J,\ \pi_I$	Ambient ADE root system Φ, internal simple vertices I, external complement J, and lattice projection $\pi_I : Q(\Phi) \to \Lambda_I = Q(\Phi)/Q(\Phi_I)$	§2.2, §2.3.1
R_β	External positive root fibre $\{\alpha \in \Phi^+ : \pi_I \alpha = \beta\}$	§2.2
$c,\ h_K,\ m$	Product $c = \prod_K c_K$ of internal component Coxeter elements of orders h_K, with cover degree $m = \operatorname{lcm}_K h_K$ $(t = s^m)$	§2.2
$\lambda(s),\ E_k$	Equivariant Cartan lift $\lambda(\zeta_m s) = c\lambda(s)$ and degree-k character eigenspace $E_k = \ker(c - \zeta_m^k)$	(2.1), §2.3.2
L	Last mandatory internal smoothness degree $\max_K(m/h_K)$ (0 if $I = \varnothing$)	§2.2.4, §2.3.2
$\Psi,\ b_{\tau,\beta},\ \ell_{\tau,\beta}$	Surviving root flat after degree-L truncation τ, exited integral contribution $b_{\tau,\beta}$, and surviving orbit size $\ell_{\tau,\beta} = \lvert R_\beta \cap \Psi^+ \rvert$	§2.2.4, Lem. 2.3
$U(n),\ \mathcal{G}(B)$	Number of unit entries $\#\{\beta \in B : n_\beta = 1\}$, and set of all realisable spectra with complete support B	§5.1, Thm. 5.15

C.2 Numerical tests and contact-atlas operators

Symbol / Term	Meaning	Location
a_i, b_i, p_{ij}, q_{ij}	Classical / BC_r multiplicities $n_{u_i}, n_{2u_i}, n_{u_i - u_j}, n_{u_i + u_j}$	§3.3
Contact class $(i \sim j)$	Equivalence class under $i = j$ or $p_{ij} > 2$ (equivalently $q_{ij} > 2$); classes with all $a_i > 4$ are *attached*, those with all $a_i = 4$ are *unattached*	Thm. 3.1
Terminal capacity κ	Permitted number of attached quadratic contact classes: $\kappa = 2$ if $S = \varnothing$ or all singleton $a_i = 2$, and $\kappa = 1$ if some singleton has $a_i > 2$	Prop. 3.5
$O(I)$	Ordinary rational root-contact test on R_I: at every attained depth k, each exiting vector avoids the $\mathbb{Q}$-span of strictly deeper vectors	Lem. 2.2, App. A
$F(I),\ s(I)$	Folded two-character test on a B_q, C_q, or G_2 core $(e = 2, 2, 3)$: $\mathbb{Q}$-span when $e \mid k$ and $\mathbb{F}_e$-span in $Q_I/\mathbb{Z}R_{I,\text{long}}$ when $e \nmid k$; $s(I)$ is the minimum short-root entry	§2.2.6, App. A
$D(a, b)$	Length-two radial condition: $b \geq 1$ and $a \in \{4, 6, \dots, 2b + 2\} \cup \{2b + 3\}$	§2.1.1, App. A
$T(a, b)$	Length-three radial condition: $(a, b) = (6, 3)$ or $a = 7,\ b \geq 4$	§2.1.1, App. A

Symbol / Term	Meaning	Location
$M(a,b,c)$	Repeated-minimum condition: $\min(a,b,c)$ is attained at least twice	§1.1, App. A
$k \mid L,\ t_1,\dots,t_m,\ x_i$	Atlas row notation: every vector in L has fixed value k; $t_1,\dots,t_m$ are the remaining positive vectors with variable entries $x_i = n(t_i)$	App. A

C.3 Terminology and support families

Term	Meaning	Location
Complete marked spectrum	Function $n : \Lambda_{\geq 0} \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ recording every nonzero genus-zero GV invariant with β and 2β kept separate and all omitted classes equal to zero	§1.1
Full support	Full positive root system of an irreducible reduced crystallographic type $(A, D, E, B, C, F_4, G_2)$ or full BC_r, with its based integral marking	§5.1
Classical frames	Terminal ($\mathbb{Z}^r$ with axes $\pm u_i$) and nonterminal (even-sum lattice without $\pm u_i$) classical supports, partitioned into singleton (S) and quadratic (M) blocks	§3.2, Prop. 3.5
Ordinary minuscule attachments	Eight families ($r \in \{4,5,6,7\}$) with internal components of distinct prime Coxeter orders ($I = A_1, A_2, A_4, A_2 + A_1$): weight fibres are fixed to 1 or 2, leaving an ordinary root core	Thm. 4.1, §6.1
Spin extensions (B_3, B_5, B_6)	Folded B_q root cores extended by spin weights in an index-two curve lattice ($13, 41, 68$ positive labels); spin entries are forced to 2 (for B_3) or 1 (for B_5, B_6)	§6.2
Diagonal B_3 and $B_4 + A_1$	Ten-label rank-three support ($B_3 \cup \{\pm(u_1+u_2+u_3)\}$, A_2 variable core) and 33-label rank-five support (E_7/A_1^2, folded B_4 core plus independent A_1)	§6.2
F_4–line attachment	50-label rank-five support (E_8/A_1^3) pairing a folded F_4 core with a length-two ray (a, b) under the single coupling $a = 4$ or $\min_s n_s = 2$	Thm. 4.1, §6.2.5
Rank-two boundary families	Five-label partial doubling, eight-label two-doubled-ray support, nine-label independent-radial support, and doubled-short G_2 two-phase support ($\epsilon \in \{4,5\}$)	§6.3
Low-rank contact atlas	The 27 signed configurations (rows 2a–2h, 3a–3k, 4a–4h) accounting for the remaining 211 exceptional marked support types in ranks $2, 3, 4$	§4.2, App. A

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